COMPUTATION OF THE HIGHEST COEFFICIENTS OF WEIGHTED EHRHART QUASI-POLYNOMIALS OF RATIONAL POLYHEDRA

V. BALDONI, N. BERLINE, J. A. DE LOERA, M. KÖPPE, AND M. VERGNE

Abstract. This article concerns the computational problem of counting the lattice points inside convex polytopes, when each point must be counted with a weight associated to it. We describe an efficient algorithm for computing the highest degree coefficients of the weighted Ehrhart quasi-polynomial for a rational simple polytope in varying dimension, when the weights of the lattice points are given by a polynomial function h. Our technique is based on a refinement of an algorithm of A. Barvinok in the unweighted case (i.e., $h \equiv 1$). In contrast to Barvinok's method, our method is local, obtains an approximation on the level of generating functions, handles the general weighted case, and provides the coefficients in closed form as step polynomials of the dilation. To demonstrate the practicality of our approach we report on computational experiments which show even our simple implementation can compete with state of the art software.

Contents

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1. Introduction

Computations with lattice points in convex polyhedra arise in various areas of computer science, mathematics, and statistics (see e.g., [13, 23, 32] and the many references therein). Given $\mathfrak{p}$, a rational convex polytope in $\mathbb{R}^d$, and $h(x)$, a polynomial function on $\mathbb{R}^d$ (often called a *weight function*), this article considers the important computational problem of computing, or estimating, the sum of the values of $h(x)$ over the lattice points belonging to $\mathfrak{p}$, namely

$$S(\mathfrak{p}, h) = \sum_{x \in \mathfrak{p} \cap \mathbb{Z}^d} h(x).$$

The function $S(\mathfrak{p}, h)$ has already been studied extensively in the *unweighted* case, i.e., when $h(x)$ takes only the constant value 1 (in that case of course $S(\mathfrak{p}, 1)$ is just the number of lattice points of $\mathfrak{p}$). Many papers and books have been written about the structure of that function (see, e.g., [11, 13] and the many references therein). Nevertheless, in many applications $h(x)$ can be a much more complicated function. Important examples of such a situation appear, for instance, in enumerative combinatorics [1], statistics [27, 22], symbolic integration [2] and non-linear optimization [24]. Still, only a small number of algorithmic results exist about the case of an arbitrary polynomial h.

It is well-known that when the polyhedron $\mathfrak{p}$ is dilated by an integer factor $n \in \mathbb{N}$, we obtain a function of n, the so-called *weighted Ehrhart quasi-polynomial* of the pair $(\mathfrak{p}, h)$, namely

$$S(n\mathfrak{p}, h) = \sum_{x \in n\mathfrak{p} \cap \mathbb{Z}^d} h(x) = \sum_{m=0}^{d+M} E_m(n \bmod q)\, n^m.$$

This is a quasi-polynomial in the sense that the function is a sum of monomials up to degree $d + M$, where $M = \deg h$, but whose coefficients E_m are periodic functions of n. The coefficient functions E_m are periodic functions with period q, where $q \in \mathbb{N}$ is the smallest positive integer such that $q\mathfrak{p}$ is a *lattice polytope*, i.e., its vertices are lattice points. We will make this more precise later (we recommend [11, 13] for excellent introductions to this topic).

To begin realizing the richness of $S(n\mathfrak{p}, h)$, note that its leading highest degree coefficient E_{d+M} (which actually does not depend on n) is precisely equal to $\int_{\mathfrak{p}} h(x)\, \mathrm{d}x$, i.e., the integral of h over the polytope $\mathfrak{p}$, when h is homogeneous of degree M. These integrals were studied in [6], [7] and more recently at [2]. Still most other coefficients are difficult to understand, even for easy polytopes, such as simplices (see [21] for a survey of results and challenges). The key aim of this article is to achieve the fast computation of the first few top-degree (weighted) coefficients E_m via an approximation of $S(n\mathfrak{p}, h)$ by a quasi-polynomial that shares the highest coefficients with $S(n\mathfrak{p}, h)$.

We now explain the known results achieved so far in the literature. It is important to stress that computing *all* the coefficients E_m for $m = 0, \dots, d+M$ is an NP-hard problem, thus the best one can hope for theoretical results is to obtain an *approximation*, as we propose to do here. Until now most results dealt only with the *unweighted case*, i.e., $h(x) = 1$ and we summarize them here: A. Barvinok first obtained for lattice polytopes $\mathfrak{p}$ a polynomial-time algorithm that for a fixed integer k_0 can compute the highest k_0 coefficients E_m (see [8]). For this he used Morelli's identities [33] and relied on an oracle that computes the volumes of faces.

Later, in [10], Barvinok obtained a formula relating the k highest degree coefficients of the (unweighted) Ehrhart quasi-polynomial of a rational polytope to volumes of sections of the polytope by certain affine lattice subspaces of codimension $< k$. As a consequence, he proved that the k highest degree coefficients of the unweighted Ehrhart quasi-polynomial of a *rational simplex* can be computed by a polynomial algorithm, when the dimension d is part of the input, but k is fixed. More precisely, given a *dilation class* $n \bmod q$ with $n \in \mathbb{N}$, Barvinok's algorithm computes the *numbers* $E_m(n \bmod q)$ by an interpolation technique. However, neither a closed formula for these E_m, depending on n, nor a generating function for the coefficients became available from [10]. In fact, in that article [10, Section 8.2] the question of efficiently computing such a closed form expression was raised.

A key point of both Barvinok's and our method is the following. The sum $S(\mathfrak{p}, h)$ has natural generalizations, the *intermediate* sums $S^L(\mathfrak{p}, h)$, where $L \subseteq V = \mathbb{R}^d$ is a rational vector subspace. For a polytope $\mathfrak{p} \subset V$ and a polynomial $h(x)$

$$S^L(\mathfrak{p}, h) = \sum_x \int_{\mathfrak{p} \cap (x+L)} h(y)\, \mathrm{d}y,$$

where the summation index x runs over the projected lattice in V/L. In other words, the polytope $\mathfrak{p}$ is sliced along lattice affine subspaces parallel to L and the integrals of h over the slices are added up. For $L = V$, there is only one term and $S^V(\mathfrak{p}, h)$ is just the integral of $h(x)$ over $\mathfrak{p}$, while, for $L = \{0\}$, we recover $S(\mathfrak{p}, h)$. Barvinok's method in [10] was to introduce particular linear combinations of the intermediate sums,

$$\sum_{L \in \mathcal{L}} \lambda(L) S^L(\mathfrak{p}, h).$$

It is natural to replace the polynomial weight $h(x)$ with an exponential function $x \mapsto \mathrm{e}^{\langle \xi, x \rangle}$, and consider the corresponding holomorphic functions of ξ in the dual V^*. Moreover, one can allow $\mathfrak{p}$ to be unbounded, then the sums

$$S^L(\mathfrak{p})(\xi) = \sum_x \int_{\mathfrak{p} \cap (x+L)} \mathrm{e}^{\langle \xi, y \rangle}\, \mathrm{d}y$$

still make sense as meromorphic functions on V^*. The map $\mathfrak{p} \mapsto S^L(\mathfrak{p})(\xi)$ is a valuation.

In [14], it was proved that a version of Barvinok's construction *on the level of generating functions*, namely $\sum_{L\in\mathcal{L}} \lambda(L)S^L(\mathfrak{p})(\xi)$, approximates $S(\mathfrak{p})(\xi)$ in a certain ring of meromorphic functions (a precise statement is given below). The proof in [14] relied on the Euler-Maclaurin expansion of these functions. Another proof, using the Poisson summation formula, will appear in [15].

In the present article, we introduce a *simplified* way to approximate $S(\mathfrak{p})(\xi)$ for the case of a simplicial affine cone $\mathfrak{p} = s + \mathfrak{c}$, which levels the way for a practical and efficient implementation. Via Brion's theorem, it is sufficient to sum up these *local* contributions of the tangent cones of the vertices. We present a method for computing the highest degree coefficients of the Ehrhart quasi-polynomial of a rational simple polytope, by applying the approximation theorem to each of the cones at vertices of $\mathfrak{p}$. The complexity depends on the number of vertices of the polytope, and thus if the simple polytope is presented by its vertices (rather than by linear inequalities), we obtain a polynomial-time algorithm. In particular, the algorithm is polynomial-time for the case of a simplex. We obtain the Ehrhart coefficient functions $E_m(n \bmod q)$ in a *closed form* as *step polynomials*, i.e., polynomials in n whose coefficients are modular expressions $(\zeta_i n) \bmod q_i$ with integers ζ_i and q_i, for example $(2n) \bmod 3$. Having a closed formula available considerably strengthens Barvinok's result in [10] even in the unweighted case $h = 1$.

The structure of this paper is as follows. In Section 2, we first present some necessary preliminaries. Section 3 explains the intermediate generating function $S^L(\mathfrak{p})(\xi)$ in more detail. Then we show how to use a grading of $S(\mathfrak{p})(\xi)$ to extract the highest degree coefficients of the weighted Ehrhart polynomial in the case of a lattice polytope. This motivates the approximation results for generating functions. In Section 4, we give a simple proof of the approximation theorem of [14], in the case of a simplicial cone (see Theorem 24). The theorem uses the notion of a *patching function* (essentially a form of Möbius inversion formulas described in Subsection 4.1). We exhibit an explicit and easily computable such patching function. Using these tools, we show in Section 5 that the approximation for a cone $s + \mathfrak{c}$ (on the level of generating functions) can be computed efficiently as a closed formula. The formula makes the periodic dependence on the vertex s explicit. Finally, in Section 6, we give the polynomial-time algorithm to compute the coefficients $E_m(n \bmod q)$ as step polynomials. Our main result (Theorem 37) says that, for every fixed number k_0, there exists a polynomial-time algorithm that, given a simple polytope $\mathfrak{p}$ of arbitrary dimension, a linear form $\ell \in V^*$, a nonnegative integer M, computes the highest $k_0 + 1$ coefficients $E_{M+d-k_0}, \dots, E_{M+d}$ of the weighted Ehrhart quasi-polynomial $S(n\mathfrak{p}, h = \ell^M)$ in the form of step polynomials.

Four comments are in order about the applicability and potential practicality of the main results: First, although the weight h used in Theorem 37 is a power of a linear form, as is carefully explained in [2], one can obtain similar complexity of computation for polynomials that depend on a fixed number of variables, or with fixed degree (Corollary 44). Second, it is also worth noting that using perturbations (see e.g., [28]), triangulations [26] or simplicial cone decompositions of polyhedra (see, e.g., [30]), one can extend computations from simple polytopes to arbitrary polytopes. Third, since our approximation is done at the level of generating functions, it extends the complexity result from [10] to the weighted case. Finally, at the end of the article we report on experiments using a simple implementation of the algorithm in *Maple*, demonstrating it is competitive with more sophisticated software tools. This indicates a potential to use this algorithm for experimentally verifying conjectures on the positivity of the Ehrhart coefficients of certain polytopes, for examples where the computation of the full Ehrhart polynomials is out of reach. The algorithms presented here require a rich mixture of computational geometry and algebraic-symbolic computation.

2. Preliminaries

2.1. **Rational convex polyhedra.** We consider a *rational vector space* V of dimension d, that is to say a finite dimensional real vector space with a lattice denoted by Λ. We will need to consider subspaces and quotient spaces of V, this is why we cannot simply let $V = \mathbb{R}^d$ and $\Lambda = \mathbb{Z}^d$. A point $v \in V$ is called *rational* if there exists a non-zero integer q such that $qv \in \Lambda$. The set of rational points in V is denoted by $V_{\mathbb{Q}}$. A subspace L of V is called *rational* if $L \cap \Lambda$ is a lattice in L. If L is a rational subspace, the image of Λ in V/L is a lattice in V/L, so that V/L is a rational vector space. The image of Λ in V/L is called the *projected lattice*. A rational space V, with lattice Λ, has a canonical Lebesgue measure $\mathrm{d}x = \mathrm{d}m_\Lambda(x)$, for which V/Λ has measure 1.

A *convex rational polyhedron* $\mathfrak{p}$ in V (we will simply say *polyhedron*) is, by definition, the intersection of a finite number of closed half spaces bounded by rational affine hyperplanes. We say that $\mathfrak{p}$ is *full-dimensional* (in V) if the affine span of $\mathfrak{p}$ is V.

In this article, a *cone* is a polyhedral cone (with vertex 0) and an affine cone is a translated set $s + \mathfrak{c}$ of a cone $\mathfrak{c}$. A cone $\mathfrak{c}$ is called *simplicial* if it is generated by independent elements of V. A simplicial cone $\mathfrak{c}$ is called *unimodular* if it is generated by independent integral vectors $v_1, \dots, v_k$ such that $\{v_1, \dots, v_k\}$ can be completed to an integral basis of V. An affine cone $\mathfrak{a}$ is called simplicial (respectively, simplicial unimodular) if the associated cone is. A *polytope* $\mathfrak{p}$ is a compact polyhedron. The set of vertices of $\mathfrak{p}$ is denoted by $\mathcal{V}(\mathfrak{p})$. For each vertex s, the cone of feasible directions at s is denoted by $\mathfrak{c}_s$. For details in all these notions see, e.g., [11].

2.2. **Generating functions: Exponential sums and integrals.**

Definition 1. *We denote by $\mathcal{H}(V^*)$ the ring of holomorphic functions defined around $0 \in V^*$. We denote by $\mathcal{M}(V^*)$ the ring of meromorphic functions defined around $0 \in V^*$ and by $\mathcal{M}_\ell(V^*) \subset \mathcal{M}(V^*)$ the subring consisting of meromorphic functions $\phi(\xi)$ which can be written as a quotient of a holomorphic function and a product of linear forms.*

This paper relies on the study of important examples of functions in $\mathcal{M}_\ell(V^*)$, the following continuous and discrete generating functions $I(\mathfrak{p}, \Lambda)$ and $S(\mathfrak{p}, \Lambda)$ associated to a convex polyhedron $\mathfrak{p}$. Both have an important additivity property which makes them valuations (see [11, Chapter 8] or the survey [12] for a detailed presentation, here we summarize the essentials).

Definition 2. *Let M be a vector space. A* valuation *F is a map from the set of polyhedra $\mathfrak{p} \subset V$ to the vector space M such that whenever the indicator functions $[\mathfrak{p}_i]$ of a family of polyhedra $\mathfrak{p}_i$ satisfy a linear relation $\sum_i r_i [\mathfrak{p}_i] = 0$, then the elements $F(\mathfrak{p}_i)$ satisfy the same relation $\sum_i r_i F(\mathfrak{p}_i) = 0$.*

Proposition 3. *There exists a unique valuation $I(\cdot, \Lambda)$ which associates to every polyhedron $\mathfrak{p} \subset V$ a meromorphic function $I(\mathfrak{p}, \Lambda) \in \mathcal{M}_\ell(V^*)$, so that the following properties hold:*

(i) *If the polyhedron $\mathfrak{p}$ is not full-dimensional or if $\mathfrak{p}$ contains a straight line, then $I(\mathfrak{p}, \Lambda) = 0$.*

(ii) *If $\xi \in V^*$ is such that $\mathrm{e}^{\langle \xi, x\rangle}$ is integrable over $\mathfrak{p}$, then*

$$I(\mathfrak{p}, \Lambda)(\xi) = \int_{\mathfrak{p}} \mathrm{e}^{\langle \xi, x\rangle} \, \mathrm{d}m_\Lambda(x).$$

(iii) *For every point $s \in V_{\mathbb{Q}}$, one has*

$$I(s + \mathfrak{p}, \Lambda)(\xi) = \mathrm{e}^{\langle \xi, s\rangle} I(\mathfrak{p}, \Lambda)(\xi).$$

We will call $I(\mathfrak{p}, \Lambda)(\xi)$ the *continuous generating function* of $\mathfrak{p}$.

Proposition 4. *There exists a unique valuation $S(\cdot, \Lambda)$ which associates to every polyhedron $\mathfrak{p} \subset V$ a meromorphic function $S(\mathfrak{p}, \Lambda) \in \mathcal{M}_\ell(V^*)$, so that the following properties hold:*

(i) *If $\mathfrak{p}$ contains a straight line, then $S(\mathfrak{p}, \Lambda) = 0$.*

(ii) *If $\xi \in V^*$ is such that $\mathrm{e}^{\langle \xi, x\rangle}$ is summable over the set of lattice points of $\mathfrak{p}$, then*

$$S(\mathfrak{p}, \Lambda)(\xi) = \sum_{x \in \mathfrak{p} \cap \Lambda} \mathrm{e}^{\langle \xi, x\rangle}.$$

(iii) *For every point $s \in \Lambda$, one has*

$$S(s + \mathfrak{p}, \Lambda)(\xi) = \mathrm{e}^{\langle \xi, s\rangle} S(\mathfrak{p}, \Lambda)(\xi).$$

$S(\mathfrak{p}, \Lambda)(\xi)$ is called the *(discrete) generating function* of $\mathfrak{p}$.

2.3. **Brion's theorem.** A consequence of the valuation property is the following fundamental theorem. It follows from the Brion-Lawrence-Varchenko decomposition of a polyhedron into the supporting cones at its vertices [18, 11]; see also [19], Proposition 3.1, for a more general Brianchon-Gram type identity.

Theorem 5. *Let $\mathfrak{p}$ be a polyhedron with set of vertices $\mathcal{V}(\mathfrak{p})$. For each vertex s, let $\mathfrak{c}_s$ be the cone of feasible directions at s. Then*

$$S(\mathfrak{p},\Lambda)=\sum_{s\in\mathcal{V}(\mathfrak{p})}S(s+\mathfrak{c}_s,\Lambda).$$

2.4. **Notations and basic facts in the case of a simplicial cone.** For all of the notions below see [11]. Let $v_i\in\Lambda, i=1,\dots,d$ be linearly independent integral vectors and let $\mathfrak{c}=\sum_{i=1}^{d}\mathbb{R}_+v_i$ be the cone they span.

Definition 6. *The* fundamental parallelepiped $\mathfrak{b}$ *of the cone (with respect to the generators $v_i, i=1,\dots,d$) is the set*

$$\mathfrak{b}=\sum_{i=1}^{d}[0,1[\,v_i.$$

Note that the set has a half-open boundary. We immediately have:

Lemma 7. *Let $s\in V$. Then*

$$I(s+\mathfrak{c},\Lambda)(\xi)=\mathrm{e}^{\langle\xi,s\rangle}\frac{(-1)^d\operatorname{vol}_\Lambda(\mathfrak{b})}{\prod_{i=1}^{d}\langle\xi,v_i\rangle},\tag{1}$$

where $\operatorname{vol}_\Lambda(\mathfrak{b})$ is the volume of the fundamental parallelepiped with respect to the Lebesgue measure $\mathrm{d}m_\Lambda$ defined by the lattice.

If $V=\mathbb{R}^d$ and $\Lambda=\mathbb{Z}^d$, then $\operatorname{vol}_\Lambda(\mathfrak{b})=|\det(v_1,\dots,v_d)|$, and so

$$I(s+\mathfrak{c},\mathbb{Z}^d)(\xi)=\mathrm{e}^{\langle\xi,s\rangle}\frac{(-1)^d|\det(v_1,\dots,v_d)|}{\prod_{i=1}^{d}\langle\xi,v_i\rangle}.\tag{2}$$

We also recall the following elementary but crucial lemma.

Lemma 8. (i) *The affine cone $(s+\mathfrak{c})\cap\Lambda$ is the disjoint union of the translated parallelepipeds $s+\mathfrak{b}+v$, for $v\in\sum_{j=1}^{d}\mathbb{N}v_j$.*

(ii) *The set of lattice points in the affine cone $s+\mathfrak{c}$ is the disjoint union of the sets $x+\sum_{i=1}^{d}\mathbb{N}v_i$ when x runs over the set $(s+\mathfrak{b})\cap\Lambda$.*

(iii) *The number of lattice points in the parallelepiped $s+\mathfrak{b}$ is equal to the volume of the parallelepiped with respect to the Lebesgue measure $\mathrm{d}m_\Lambda$ defined by the lattice, that is*

$$\operatorname{Card}((s+\mathfrak{b})\cap\Lambda)=\operatorname{vol}_\Lambda(\mathfrak{b}).$$

In particular, when $V=\mathbb{R}^d$ and $\Lambda=\mathbb{Z}^d$, then

$$\operatorname{Card}((s+\mathfrak{b})\cap\mathbb{Z}^d)=|\det(v_1,\dots,v_d)|.$$

The study of the generating function $S(s+\mathfrak{c},\Lambda)(\xi)$ of the affine cone $s+\mathfrak{c}$ will be a crucial tool. It relies on expressing $S(s+\mathfrak{c},\Lambda)(\xi)$ in terms of the generating function $S(s+\mathfrak{b},\Lambda)(\xi)=\sum_{x\in(s+\mathfrak{b})\cap\Lambda}\mathrm{e}^{\langle\xi,x\rangle}$ of the fundamental parallelepiped. Lemma 8 (ii) immediately gives:

Lemma 9.

$$S(s+\mathfrak{c},\Lambda)(\xi)=S(s+\mathfrak{b},\Lambda)(\xi)\frac{1}{\prod_{j=1}^{d}(1-\mathrm{e}^{\langle\xi,v_j\rangle})}. \tag{3}$$

Example 10. Consider the case where $V=\mathbb{R}$ and $\Lambda=\mathbb{Z}$. Let $\mathfrak{c}=\mathbb{R}_+$. Let $s\in\mathbb{R}$, then the "fractional part" $\{s\}\in[0,1[$ is defined as the unique real number such that $s-\{s\}\in\mathbb{Z}$. Then the unique integer $\bar{s}$ in $s+\mathfrak{b}$ is $s+\{-s\}$, and so equations (1) and (3) give

$$I(s+\mathfrak{c},\mathbb{Z})(\xi)=\mathrm{e}^{\xi s}\frac{-1}{\xi}\quad\text{and}\quad S(s+\mathfrak{c},\mathbb{Z})(\xi)=\mathrm{e}^{\xi(s+\{-s\})}\frac{1}{1-\mathrm{e}^{\xi}}.$$

3. Key ideas of the approximation theory

3.1. **Weighted Ehrhart quasi-polynomials.** Let $\mathfrak{p}\subset V$ be a rational polytope and let $h(x)$ be a polynomial function of degree M on V. We consider the following weighted sum over the set of lattice points of $\mathfrak{p}$,

$$\sum_{x\in\mathfrak{p}\cap\Lambda}h(x).$$

When $\mathfrak{p}$ is dilated by a non-negative integer $n\in\mathbb{N}$, we obtain the weighted Ehrhart quasi-polynomial of the pair $(\mathfrak{p},h)$.

Definition 11. *Let q be the smallest positive integer such that $q\mathfrak{p}$ is a lattice polytope. The we define the Ehrhart quasi-polynomial $E(\mathfrak{p},h;n)$ and its coefficients $E_m(\mathfrak{p},h;n \bmod q)$ by*

$$E(\mathfrak{p},h;n)=\sum_{x\in n\mathfrak{p}\cap\Lambda}h(x)=\sum_{m=0}^{d+M}E_m(\mathfrak{p},h;n \bmod q)\,n^m.$$

We note that the coefficients E_m depend on n, but they actually depend only on $n \bmod q$, where q is the smallest positive integer such that $q\mathfrak{p}$ is a lattice polytope. If $h(x)$ is homogeneous of degree M, the highest degree coefficient E_{d+M} is equal to the integral $\int_{\mathfrak{p}}h(x)\,\mathrm{d}x$ (see [2] and references therein).

We concentrate on the special case where the polynomial $h(x)$ is a power of a linear form

$$h(x)=\frac{\langle\xi,x\rangle^M}{M!}.$$

This is not a restriction because any polynomial can be written as a linear combination of powers of linear forms. In fact, as discussed in [2], whenever the polynomial $h(x)$ is either of fixed degree or only depends on a fixed number of variables (possibly after a linear change of variables), then only a

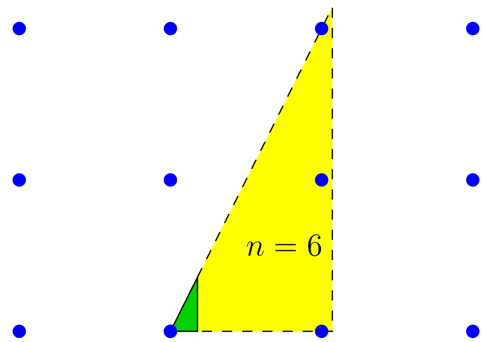

FIGURE 1. The example triangle $\mathfrak{t}$ and its dilation $6\mathfrak{t}$

polynomial number of powers of linear forms are needed, and such a decomposition can be computed in polynomial time. We introduce the following notation.

Definition 12. *Let q be as above. We define the Ehrhart quasi-polynomial $E(\mathfrak{p}, \xi, M; n)$ and the coefficients $E_m(\mathfrak{p}, \xi, M; n \bmod q)$ for $m = 0, \dots, M+d$ by*

$$E(\mathfrak{p}, \xi, M; n) = \sum_{x \in n\mathfrak{p} \cap \Lambda} \frac{\langle \xi, x \rangle^M}{M!} = \sum_{m=0}^{M+d} E_m(\mathfrak{p}, \xi, M; n \bmod q)\, n^m.$$

It will be convenient in this paper to introduce the following notations. For a positive integer $q \in \mathbb{N}$ and a real number $n \in \mathbb{R}$, we write

$$\lfloor n \rfloor_q := q \left\lfloor \tfrac{1}{q} n \right\rfloor \in q\mathbb{Z}, \quad \{n\}_q := (n \bmod q) \in [0, q),$$

which give the unique decomposition

$$n = \lfloor n \rfloor_q + \{n\}_q.$$

By $\lfloor n \rfloor := \lfloor n \rfloor_1$ and $\{n\} := \{n\}_1$ we obtain the ordinary "floor" and "fractional part" notations. Finally, $\lceil n \rceil := -\lfloor -n \rfloor$ is the "ceiling" notation.

Example 13. Consider the rational triangle $\mathfrak{t}$ with vertices $(0, 0)$, $(\frac{5}{28}, 0)$, and $(\frac{5}{28}, \frac{5}{14})$ as shown in Figure 1. Let us compute $E(\mathfrak{t}, \xi, M; n)$ for this small example. Note that the integer q such that $q\mathfrak{t}$ is a lattice polytope is $q = 28$.

In what follows consider powers of the linear form $\xi = x + y$ as weights for the lattice points. When the power $M = 0$, then we obtain a constant weight and the quasi-polynomial $E(\mathfrak{t}, \xi, 0; n)$ counts the lattice points inside the various dilations of $\mathfrak{t}$: it is given by the following formula:

$$\frac{25}{784} n^2 + \left(-\frac{5}{392} \{5\,n\}_{28} + \frac{5}{14} \right) n + \left(1 + \frac{1}{784} \left(\{5\,n\}_{28}\right)^2 - \frac{1}{14} \{5\,n\}_{28} \right).$$

Indeed, when $n = 1$ (no dilation) there is only one lattice point and the formula above reduces to

$$\frac{1089}{784} - \frac{33}{392} \{5\}_{28} + \frac{1}{784} \left(\{5\}_{28}\right)^2 = \frac{1089}{784} - \frac{33}{392}\, 5 + \frac{1}{784}\, 25 = 1.$$

When we dilate the same triangle six times, i.e., $n = 6$, we obtain four lattice points.

$$\frac{841}{196} - \frac{29}{196}\{30\}_{28} + \frac{1}{784}(\{30\}_{28})^2 = \frac{841}{196} - \frac{29}{196}\cdot 2 + \frac{1}{784}\cdot 4 = 4.$$

Next let us take $M = 1$, in that case the lattice point (a, b) is counted with weight $a + b$. In this case the top coefficient is equal to the integral of the linear form $\xi = x + y$ over $\mathfrak{t}$. The quasi-polynomial $E(\mathfrak{t}, \xi, 1; n)$ is given by the following formula:

$$\frac{125}{16464}n^3 + \left(-\frac{25}{5488}\{5\,n\}_{28} + \frac{75}{784}\right)n^2 + \left(\frac{5}{5488}(\{5\,n\}_{28})^2 - \frac{15}{392}\{5\,n\}_{28} + \frac{25}{84}\right)n$$
$$+\left(-\frac{1}{16464}(\{5\,n\}_{28})^3 + \frac{3}{784}(\{5\,n\}_{28})^2 - \frac{5}{84}\{5\,n\}_{28}\right).$$

Substitute again $n = 1$ in the expression, to obtain

$$\frac{275}{686} - \frac{1685}{16464}\{5\}_{28} + \frac{13}{2744}(\{5\}_{28})^2 - \frac{1}{16464}(\{5\}_{28})^3 = 0.$$

Note that since only the lattice point $(0, 0)$ lies within the triangle at $n = 1$, the quasipolynomial must evaluate to zero.

In practice, it is impossible to compute $E(\mathfrak{p}, \xi, M; n)$ except when $\mathfrak{p}$ is of small dimension (and M relatively small). Thus we restrict our ambitions:

Let us fix a number k_0. Our goal will be to compute the $k_0 + 1$ highest degree coefficients $E_m(\mathfrak{p}, \xi, M; n \bmod q)$, for $m = M + d, \dots, M + d - k_0$. We will be able to give a polynomial time algorithm to do so.

3.2. **Grading of the generating functions.** A key property that we will make use of in this article is the following *grading* of $\mathcal{M}_\ell$: A function $\phi(\xi) \in \mathcal{M}_\ell(V^*)$ has a unique expansion into homogeneous rational functions

$$\phi(\xi) = \sum_{m \geq m_0} \phi_{[m]}(\xi),$$

where the summands $\phi_{[m]}(\xi)$ have degree m as we define now: If P is a homogeneous polynomial on V^* of degree p, and D a product of r linear forms, then $\frac{P}{D}$ is an element in $\mathcal{M}_\ell(V^*)$ homogeneous of degree $m = p - r$. For instance, $\frac{\xi_1}{\xi_2}$ is homogeneous of degree 0. On this example we observe that a function in $\mathcal{M}_\ell(V^*)$ which has non-negative degree terms need not be analytic.

In particular, consider the generating function $S(s + \mathfrak{c}, \Lambda)(\xi)$ for a simplicial cone $\mathfrak{c}$. By Lemma 9,

$$S(s + \mathfrak{c}, \Lambda)(\xi) = S(s + \mathfrak{b}, \Lambda)(\xi)\frac{1}{\prod_{j=1}^{d}(1 - \mathrm{e}^{\langle \xi, v_j \rangle})}.$$

Thus, $S(s + \mathfrak{c}, \Lambda) \in \mathcal{M}_\ell(V^*)$, and so it admits a decomposition into homogeneous components:

Lemma 14.

$$S(s+\mathfrak{c},\Lambda)(\xi)=S(s+\mathfrak{c},\Lambda)_{[-d]}(\xi)+S(s+\mathfrak{c},\Lambda)_{[-d+1]}(\xi)+\cdots,\qquad(4)$$

and the lowest degree term $S(s+\mathfrak{c},\Lambda)_{[-d]}(\xi)$ is equal to $I(\mathfrak{c},\Lambda)(\xi)$, i.e., the integral over the unshifted cone $\mathfrak{c}$.

Proof. We write

$$\prod_{j=1}^{d}\frac{1}{1-\mathrm{e}^{\langle\xi,v_j\rangle}}=\prod_{j=1}^{d}\frac{\langle\xi,v_j\rangle}{1-\mathrm{e}^{\langle\xi,v_j\rangle}}\frac{1}{\prod_{j=1}^{d}\langle\xi,v_j\rangle}.\qquad(5)$$

The function $\frac{x}{1-\mathrm{e}^{x}}$ is holomorphic with value -1 for $x=0$. Thus we have $S(s+\mathfrak{c},\Lambda)\in\mathcal{M}_\ell(V^*)$. The value at $\xi=0$ of the sum over the parallelepiped is the number of lattice points of the parallelepiped, that is $\mathrm{vol}_\Lambda(\mathfrak{b})$. This proves the last assertion. □

3.3. **Sketch of the method for lattice polytopes.** We will now explain the key point of our method, with the simplifying assumption that the vertices of the polytope are lattice points. We will show that the highest degree coefficients of the weighted Ehrhart polynomial can be read out from an approximation of the generating functions of the cones at vertices. In Section 4 we will study this approximation, and in Section 5 we will show how to efficiently compute it. Then, in Section 6, we will come back to the computation of Ehrhart coefficients for the general case of rational polytopes.

Proposition 15. *Let $\mathfrak{p}$ be a lattice polytope. Then, for $k\geq 0$, we have*

$$E_{M+d-k}(\mathfrak{p},\xi,M)=\sum_{s\in\mathcal{V}(\mathfrak{p})}\frac{\langle\xi,s\rangle^{M+d-k}}{(M+d-k)!}S(\mathfrak{c}_s)_{[-d+k]}(\xi).\qquad(6)$$

The highest degree coefficient is just the integral

$$E_{M+d}(\mathfrak{p},\xi,M)=\int_{\mathfrak{p}}\frac{\langle\xi,x\rangle^{M}}{M!}\,\mathrm{d}x.$$

Remark 16. *As functions of ξ, the coefficients $E_m(\mathfrak{p},\xi,M)$ are polynomial, homogeneous of degree M. However, in (6), they are expressed as linear combinations of rational functions of ξ, whose poles cancel out.*

Proof of Proposition 15. The starting point is Brion's formula. As the vertices are lattice points, we have

$$\sum_{x\in\mathfrak{p}\cap\Lambda}\mathrm{e}^{\langle\xi,x\rangle}=\sum_{s\in\mathcal{V}(\mathfrak{p})}S(s+\mathfrak{c}_s)(\xi)=\sum_{s\in\mathcal{V}(\mathfrak{p})}\mathrm{e}^{\langle\xi,s\rangle}S(\mathfrak{c}_s)(\xi).\qquad(7)$$

When $\mathfrak{p}$ is replaced with $n\mathfrak{p}$, the vertex s is replaced with ns but the cone $\mathfrak{c}_s$ does not change. We obtain

$$\sum_{x\in n\mathfrak{p}\cap\Lambda}\mathrm{e}^{\langle\xi,x\rangle}=\sum_{s\in\mathcal{V}(\mathfrak{p})}\mathrm{e}^{n\langle\xi,s\rangle}S(\mathfrak{c}_s)(\xi).$$

We replace ξ with $t\xi$,

$$\sum_{x\in n\mathfrak{p}\cap\Lambda} \mathrm{e}^{t\langle\xi,x\rangle} = \sum_{s\in\mathcal{V}(\mathfrak{p})} \mathrm{e}^{nt\langle\xi,s\rangle} S(\mathfrak{c}_s)(t\xi).$$

The decomposition into homogeneous components gives

$$S(\mathfrak{c}_s)(t\xi) = t^{-d} I(\mathfrak{c}_s)(\xi) + t^{-d+1} S(\mathfrak{c}_s)_{[-d+1]}(\xi) + \cdots + t^k S(\mathfrak{c}_s)_{[k]}(\xi) + \cdots .$$

Hence, the t^M-term in the right-hand side of the above equation is equal to

$$\sum_{k=0}^{M+d} (nt)^{M+d-k}\, t^{-d+k} \frac{\langle\xi,s\rangle^{M+d-k}}{(M+d-k)!} S(\mathfrak{c}_s)_{[-d+k]}(\xi).$$

Thus we have

$$\begin{aligned}\sum_{x\in n\mathfrak{p}\cap\Lambda} \frac{\langle\xi,x\rangle^M}{M!} = \sum_{s\in\mathcal{V}(\mathfrak{p})} & n^{M+d} \frac{\langle\xi,s\rangle^{M+d}}{(M+d)!} I(\mathfrak{c}_s)(\xi) \\ & + n^{M+d-1} \frac{\langle\xi,s\rangle^{M+d-1}}{(M+d-1)!} S(\mathfrak{c}_s)_{[-d+1]}(\xi) + \cdots + S(\mathfrak{c}_s)_{[M]}(\xi). \end{aligned} \tag{8}$$

From this relation, we read immediately that $\sum_{x\in n\mathfrak{p}\cap\Lambda} \frac{\langle\xi,x\rangle^M}{M!}$ is a polynomial function of n of degree $M+d$, and that the coefficient of n^{M+d-k} is given by (6). The highest degree coefficient is given by

$$E_{M+d}(\mathfrak{p},\xi,M) = \sum_{s\in\mathcal{V}(\mathfrak{p})} \frac{\langle\xi,s\rangle^{M+d}}{(M+d)!} I(\mathfrak{c}_s)(\xi).$$

Applying Brion's formula for the integral, this is equal to the term of ξ-degree M in $I(\mathfrak{p})(\xi)$, which is indeed the integral $\int_{\mathfrak{p}} \frac{\langle\xi,x\rangle^M}{M!}\,\mathrm{d}x$. □

From Proposition 15, we draw an important consequence: in order to compute the k_0+1 highest degree terms of the weighted Ehrhart polynomial for the weight $h(x) = \frac{\langle\xi,x\rangle^M}{M!}$, we *only need* the k_0+1 lowest degree homogeneous terms of the meromorphic function $S(\mathfrak{c}_s)(\xi)$, for every vertex s of $\mathfrak{p}$. We compute such an approximation in Section 4; it turns out to be sufficient also in the general case of a rational polytope.

3.4. **Intermediate generating functions.** To obtain the approximation, we study generating functions which interpolate between the integral $I(\mathfrak{p},\Lambda)$ and the discrete sum $S(\mathfrak{p},\Lambda)$. This trend of ideas was first discussed by Barvinok in [10]. Let L be a rational subspace of V. To any polyhedron $\mathfrak{p}$ we associate a meromorphic function $S^L(\mathfrak{p},\Lambda)(\xi) \in \mathcal{M}(V^*)$, which is, roughly speaking, obtained by slicing $\mathfrak{p}$ along affine subpaces parallel to L through lattice points, and adding the integrals of $\mathrm{e}^{\langle\xi,x\rangle}$ along the slices. Recall that the quotient space V/L is endowed with the projected lattice $\Lambda_{V/L}$.

Proposition 17. *Let $L \subseteq V$ be a rational subspace. There exists a unique valuation $S^L(\cdot, \Lambda)$ which to every rational polyhedron $\mathfrak{p} \subset V$ associates a meromorphic function with rational coefficients $S^L(\mathfrak{p}, \Lambda) \in \mathcal{M}(V^*)$ so that the following properties hold:*

(i) *If $\mathfrak{p}$ contains a line, then $S^L(\mathfrak{p}, \Lambda) = 0$.*

(ii)
$$S^L(\mathfrak{p}, \Lambda)(\xi) = \sum_{x \in \Lambda_{V/L}} \int_{\mathfrak{p} \cap (x+L)} \mathrm{e}^{\langle \xi, y \rangle} \, \mathrm{d}y, \tag{9}$$
for every $\xi \in V^*$ such that the above sum converges.

(iii) *For every point $s \in \Lambda$, we have*
$$S^L(s + \mathfrak{p}, \Lambda)(\xi) = \mathrm{e}^{\langle \xi, s \rangle} S^L(\mathfrak{p}, \Lambda)(\xi).$$

We call the function $S^L(\mathfrak{p}, \Lambda)$ an *intermediate generating function*. The proof is entirely analogous to the case $L = \{0\}$, see Theorem 3.1 in [12], and we omit it.

For $L = \{0\}$, we recover the valuation S. For $L = V$, we have $S^V(\mathfrak{p}, \Lambda) = I(\mathfrak{p}, \Lambda)$. In particular, if $\mathfrak{p}$ is not full-dimensional, then $S^V(\mathfrak{p}, \Lambda) = 0$.

If $\mathfrak{p}$ is compact, the meromorphic function $S^L(\mathfrak{p}, \Lambda)(\xi)$ is actually regular at $\xi = 0$, and its value for $\xi = 0$ is the $\mathbb{Q}$-valued valuation $E_{L^\perp}(\mathfrak{p})$ considered by Barvinok [10].

Remark 18. *The function $S^L(\mathfrak{p}, \Lambda)$ is actually an element of $\mathcal{M}_\ell(V^*)$, just like the functions $S(\mathfrak{p}, \Lambda)$ and $I(\mathfrak{p}, \Lambda)$. This follows from an interesting decomposition that allows to write S^L as a combination of terms using S and I for certain cones. This and other properties of the valuation $S^L(\cdot, \Lambda)$ will be discussed in a forthcoming article* [4].

4. Approximation of the generating function of a simplicial affine cone

Let $\mathfrak{c} \subset V$ be a simplicial cone with integral generators v_j, $j = 1, \dots, d$, and let $s \in V_{\mathbb{Q}}$. Let $k_0 \le d$. In this section we will obtain an expression for the $k_0 + 1$ lowest degree homogeneous terms of the meromorphic function $S(s + \mathfrak{c})(\xi)$. Recall that if $\mathfrak{c}$ is unimodular, the function $S(s + \mathfrak{c})(\xi)$ has a "short" expression
$$S(s + \mathfrak{c})(\xi) = \mathrm{e}^{\langle \xi, \bar{s} \rangle} \prod_{j=1}^{d} \frac{1}{1 - \mathrm{e}^{\langle \xi, v_j \rangle}},$$
where v_i, $i = 1, \dots, d$ are the primitive integral generators of the edges and $\bar{s}$ is the unique lattice point in the corresponding parallelepiped $s + \mathfrak{b}$. This is a particular case of Lemma 8.

When $\mathfrak{c}$ is not unimodular, it is not possible to compute efficiently the first k_0 terms of the Laurent expansion of the function $S(s + \mathfrak{c})(\xi)$, if k_0 is part of the input as well as the dimension d. In contrast, *if k_0 is fixed*, we are going to obtain an expression for the terms of degree $\le -d + k_0$

which only involves a discrete summation over cones in dimension $\leq k_0$ and determinants. For example, the lowest degree term is $|\det(v_j)| \prod_j \frac{-1}{\langle \xi, v_j \rangle}$.

4.1. **Patching functions.** For constructing the approximation, we will use a *patching function.* For $I \subseteq \{1, \dots, d\}$, we denote by L_I the linear span of the vectors v_i, $i \in I$ and by $L_I^\perp \subseteq V^*$ the orthogonal subspace. We denote by I^c the complement of I in $\{1, \dots, d\}$.

Definition 19. *We denote by $\mathcal{J}^d_{\geq d_0}$ the set of subsets $I \subseteq \{1, \dots, d\}$ of cardinality $|I| \geq d_0$. A function $I \mapsto \lambda(I)$ on $\mathcal{J}^d_{\geq d_0}$ is called a* patching function *if it satisfies the following condition.*

$$\Big[\bigcup_{I \in \mathcal{J}^d_{\geq d_0}} L_I^\perp \Big] = \sum_{I \in \mathcal{J}^d_{\geq d_0}} \lambda(I) \big[L_I^\perp \big]. \tag{10}$$

Remark 20. *The family of subspaces L_I, $|I| \geq d_0$ is closed under sum, and the family of orthogonals $L_I^\perp$ is closed under intersection. The value $\lambda(I)$ plays the same as role as the Möbius function $\mu(L)$ for $L_I^\perp$ that Barvinok* [10, section 7] *computes algorithmically for a certain family of subspaces L by walking the poset. From this discussion, it follows that patching functions do exist. The precise relation between Barvinok's construction and the construction of the present paper will be studied in the forthcoming paper* [5].

We will compute a canonical patching function below, in Proposition 27. Let us state some interesting properties.

Lemma 21. *Let $I \mapsto \lambda(I)$ be a function on $\mathcal{J}^d_{\geq d_0}$. The following conditions are equivalent.*

(i) *λ is a patching function.*
(ii) *$\sum_{I \in \mathcal{J}^d_{\geq d_0}, I \subseteq I_0} \lambda(I) = 1$ for every $I_0 \in \mathcal{J}^d_{\geq d_0}$.*
(iii) *For $1 \leq i \leq d$, let $F_i(z) \in \mathbb{C}[[z]]$ be a formal power series (in one variable) with constant term equal to 1. Then*

$$\prod_{1 \leq i \leq d} F_i(z_i) \equiv \sum_{I \in \mathcal{J}^d_{\geq d_0}} \lambda(I) \prod_{i \in I^c} F_i(z_i) \quad \textit{mod terms of } z\textit{-degree } \geq d - d_0 + 1. \tag{11}$$

(iv) *Let $z_{I^c} = \sum_{i \in I^c} z_i$. Then*

$$\mathrm{e}^{z_1 + \dots + z_d} \equiv \sum_{I \in \mathcal{J}^d_{\geq d_0}} \lambda(I) \mathrm{e}^{z_{I^c}} \textit{ mod terms of } z\textit{-degree } \geq d - d_0 + 1. \tag{12}$$

Proof. Let $I_0 \in \mathcal{J}^d_{\geq d_0}$. Then there exists $\xi \in L_{I_0}^\perp$ such that $\xi \in L_I^\perp$ if and only if $L_{I_0}^\perp \subseteq L_I^\perp$, i.e., if and only if $I \subseteq I_0$. Thus (i) $\Leftrightarrow$ (ii).

Let us prove that (ii) $\Rightarrow$ (iii). We write $F_i(z_i) = 1 + z_i g_i(z_i)$. We have

$$\prod_{1 \leq i \leq d} (1 + z_i g_i(z_i)) = \sum_{K \subseteq \{1, \dots, d\}} \prod_{i \in K} z_i g_i(z_i). \tag{13}$$

Consider a monomial $z_1^{k_1}\cdots z_d^{k_d}$ of total degree $k_1+\cdots+k_d\leq d-d_0$. Let us denote its coefficient in the product $\prod_{i\in K} z_i g_i(z_i)$ by α_K. Let K_0 be the set of indices such that $k_i\neq 0$. Then $|K_0|\leq d-d_0$. Moreover in the right hand side of (13), our monomial appears only in the terms where $K\subseteq K_0$. Therefore the coefficient of $z_1^{k_1}\cdots z_d^{k_d}$ in $\prod_{1\leq i\leq d} F_i(z_i)$ is equal to $\sum_{K\subseteq K_0}\alpha_K$. Furthermore, the coefficient of $z_1^{k_1}\cdots z_d^{k_d}$ in the right-hand-side of (11) is equal to

$$\sum_{I\in\mathcal{J}^d_{\geq d_0}}\lambda(I)\sum_{K\subseteq K_0\cap I^c}\alpha_K=\sum_{K\subseteq K_0}\alpha_K\sum_{\substack{I\in\mathcal{J}^d_{\geq d_0}\\ K\subseteq I^c}}\lambda(I).$$

By condition (ii) we have

$$\sum_{\substack{I\in\mathcal{J}^d_{\geq d_0}\\ K\subseteq I^c}}\lambda(I)=1\quad\text{for every } K\subseteq K_0.$$

Thus we have proved that (ii) $\Rightarrow$ (iii). Next, (iv) is a particular case of (iii), so it remains only to prove that (iv) implies (ii).

By expanding the exponentials in condition (iv), we obtain

$$(z_1+\cdots+z_d)^{d-d_0}=\sum_{I\in\mathcal{J}^d_{\geq d_0}}\lambda(I)\Big(\sum_{i\in I^c} z_i\Big)^{d-d_0}.$$

Condition (ii) follows easily from this relation. $\square$

4.2. **Formula for intermediate sums.** In preparation for the approximation theorem, we need some notations and an expression for intermediate sums $S^L(s+\mathfrak{c},\Lambda)(\xi)$.

We have $V=L_I\oplus L_{I^c}$. For $x\in V$ we denote the components by

$$x=x_I+x_{I^c}.$$

Thus we identify the quotient V/L_I with L_{I^c} and we denote the projected lattice by $\Lambda_{I^c}\subset L_{I^c}$. Note that $L_{I^c}\cap\Lambda\subseteq\Lambda_{I^c}$, but the inclusion is strict in general.

Example 22. Let $v_1=(1,0), v_2=(1,2)$, $I=\{2\}$. The projected lattice Λ_{I^c} on $L_{I^c}=\mathbb{R}v_1$ is $\mathbb{Z}\frac{v_1}{2}$. See Figure 2.

We denote by $\mathfrak{c}_I$ the cone generated by the vectors v_j, for $j\in I$ and by $\mathfrak{b}_I$ the parallelepiped $\mathfrak{b}_I=\sum_{i\in I}[0,1[\,v_i$. Similarly we denote by $\mathfrak{c}_{I^c}$ the cone generated by the vectors v_j, for $j\in I^c$ and $\mathfrak{b}_{I^c}=\sum_{i\in I^c}[0,1[\,v_i$. The projection of the cone $\mathfrak{c}$ on $V/L_I=L_{I^c}$ identifies with $\mathfrak{c}_{I^c}$. Note that the generators v_i, $i\in I^c$, may be non-primitive for the projected lattice Λ_{I^c}, even if it is primitive for Λ, as we see in the previous example. We write $s=s_I+s_{I^c}$.

We first show that the intermediate generating function $S^{L_I}(s+\mathfrak{c},\Lambda)$ decomposes as a product.

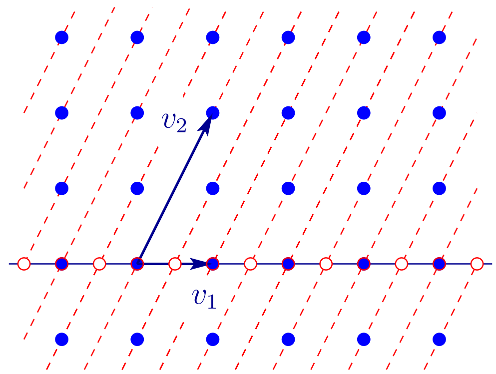

FIGURE 2. The projected lattice $\Lambda_{\{1\}}$

The function $S(s_{I^c} + \mathfrak{c}_{I^c}, \Lambda_{I^c})(\xi)$ is a meromorphic function on the space $(L_{I^c})^*$. The integral $I(s_I + \mathfrak{c}_I, L_I \cap \Lambda)(\xi)$ is a meromorphic function on the space $(L_I)^*$. We consider both as functions on V^* through the decomposition $V = L_I \oplus L_{I^c}$.

Proposition 23. *The intermediate sum for the full cone $s + \mathfrak{c}$ breaks up into the product*

$$S^{L_I}(s + \mathfrak{c}, \Lambda)(\xi) = S(s_{I^c} + \mathfrak{c}_{I^c}, \Lambda_{I^c})(\xi)\, I(s_I + \mathfrak{c}_I, L_I \cap \Lambda)(\xi). \tag{14}$$

Proof. The projection of the cone $s+\mathfrak{c}$ into L_{I^c} is the cone $s_{I^c}+\mathfrak{c}_{I^c}$. For each $x_{I^c} \in (s_{I^c} + \mathfrak{c}_{I^c}) \cap \Lambda_{I^c}$, the slice $(s + \mathfrak{c}) \cap (x_{I^c} + L_I)$ is the cone $x_{I^c} + s_I + \mathfrak{c}_I$. Let us compute the integral on the slice.

$$\int_{(s+\mathfrak{c})\cap(x_{I^c}+L_I)} \mathrm{e}^{\langle \xi, y\rangle}\, \mathrm{d}m_{L_I\cap\Lambda}(y). \tag{15}$$

We write $y = x_{I^c} + s_I + \sum_{j\in I} y_j v_j$. Then

$$\mathrm{d}m_{L_I\cap\Lambda}(y) = \mathrm{vol}_{L_I\cap\Lambda}(\mathfrak{b}_I) \prod_{j\in I} \mathrm{d}y_j.$$

Hence (15) is equal to

$$\mathrm{e}^{\langle \xi, x_{I^c}\rangle} \mathrm{e}^{\langle \xi, s_I\rangle}\, \mathrm{vol}_{L_I\cap\Lambda}(\mathfrak{b}_I)(-1)^{|I|} \prod_{j\in I} \frac{1}{\langle \xi, v_j\rangle}.$$

We observe that only the first factor, $\mathrm{e}^{\langle \xi, x_{I^c}\rangle}$, depends on x_{I^c}. The sum of these factors over all $x_{I^c} \in (s_{I^c} + \mathfrak{c}_{I^c}) \cap \Lambda_{I^c}$ gives $S(s_{I^c} + \mathfrak{c}_{I^c}, \Lambda_{I^c})(\xi)$, and using formula (1) for the integral, we obtain (14). □

4.3. **Approximation theorem.** We can now state and prove the approximation theorem.

Theorem 24 (Approximation by a patched generating function)**.** *Let $\mathfrak{c} \subset V$ be a rational simplicial cone with edge generators $v_1, \dots, v_d$. Let $s \in V_{\mathbb{Q}}$. Let*

$I \mapsto \lambda(I)$ be a patching function on $\mathcal{J}^d_{\geq d_0}$. For $I \in \mathcal{J}^d_{\geq d_0}$ let L_I be the linear span of $\{v_i\}_{i\in I}$. Then we have

$$S(s+\mathfrak{c},\Lambda)(\xi) \equiv A^\lambda(s+\mathfrak{c},\Lambda)(\xi) := \sum_{I\in\mathcal{J}^d_{\geq d_0}} \lambda(I)\, S^{L_I}(s+\mathfrak{c},\Lambda)(\xi)$$
$$\textit{mod terms of } \xi\textit{-degree } \geq -d_0+1. \quad (16)$$

We call the function $A^\lambda(s+\mathfrak{c},\Lambda)(\xi)$ on the right-hand side of (16) the *patched generating function* of $s+\mathfrak{c}$ (with respect to λ).

Proof of Theorem 24. We write the vertex as $s=\sum_i s_i v_i$. Let $a=\sum_i a_i v_i \in V$. We apply (11) to the functions

$$F_i(z_i) = \mathrm{e}^{(a_i-s_i)z_i}\frac{-z_i}{1-\mathrm{e}^{z_i}},$$

and we substitute $z_i=\langle\xi,v_i\rangle$. We obtain

$$\mathrm{e}^{\langle\xi,a-s\rangle}\prod_{i=1}^d \frac{-\langle\xi,v_i\rangle}{1-\mathrm{e}^{\langle\xi,v_i\rangle}} \equiv \sum_{I\in\mathcal{J}^d_{\geq d_0}} \lambda(I)\mathrm{e}^{\langle\xi,a_{I^c}-s_{I^c}\rangle}\prod_{i\in I^c}\frac{-\langle\xi,v_i\rangle}{1-\mathrm{e}^{\langle\xi,v_i\rangle}}$$
$$\text{mod terms of } \xi\text{-degree } \geq d-d_0+1.$$

We multiply both sides first by $\mathrm{e}^{\langle\xi,s\rangle}$; because this is analytic in ξ and thus of non-negative ξ-degree, the identity modulo terms of high ξ-degree still holds true. Then we multiply by $1/\prod_{i=1}^d(-\langle\xi,v_i\rangle)$, which is homogeneous of degree $-d$ in ξ. We obtain

$$\mathrm{e}^{\langle\xi,a\rangle}\prod_{i=1}^d\frac{1}{1-\mathrm{e}^{\langle\xi,v_i\rangle}} \equiv \sum_{I\in\mathcal{J}^d_{\geq d_0}}\lambda(I)\mathrm{e}^{\langle\xi,a_{I^c}\rangle}\prod_{i\in I^c}\frac{1}{1-\mathrm{e}^{\langle\xi,v_i\rangle}}\mathrm{e}^{\langle\xi,s_I\rangle}\prod_{i\in I}\frac{-1}{\langle\xi,v_i\rangle}$$
$$\text{mod terms of } \xi\text{-degree } \geq -d_0+1. \quad (17)$$

Now, we sum up equalities (17) when a runs over the set $(s+\mathfrak{b})\cap\Lambda$ of integral points in the fundamental parallelepiped $s+\mathfrak{b}$ of the affine cone $s+\mathfrak{c}$. On the left-hand side we obtain

$$\sum_{a\in(s+\mathfrak{b})\cap\Lambda}\mathrm{e}^{\langle\xi,a\rangle}\prod_{i=1}^d\frac{1}{1-\mathrm{e}^{\langle\xi,v_i\rangle}}.$$

By Lemma 9, this is precisely $S(s+\mathfrak{c},\Lambda)(\xi)$. On the right-hand side, for each I, we have a sum over $a\in(s+\mathfrak{b})\cap\Lambda$ of the function

$$\mathrm{e}^{\langle\xi,a_{I^c}\rangle}\prod_{i\in I^c}\frac{1}{1-\mathrm{e}^{\langle\xi,v_i\rangle}}\mathrm{e}^{\langle\xi,s_I\rangle}\prod_{i\in I}\frac{-1}{\langle\xi,v_i\rangle},$$

which depends only on the projection a_{I^c} of a in the decomposition $a = a_I + a_{I^c} \in L_I \oplus L_{I^c}$. When a runs over $(s+\mathfrak{b})\cap\Lambda$, its projection a_{I^c} runs over $(s_{I^c}+\mathfrak{b}_{I^c})\cap\Lambda_{I^c}$. Let us show that the fibers have the same number of points, equal to $\mathrm{vol}_{L_I\cap\Lambda}(\mathfrak{b}_I)$. For a given $a_{I^c}\in(s_{I^c}+\mathfrak{b}_{I^c})\cap\Lambda_{I^c}$, let us compute the fiber

$$\{\, y\in(s+\mathfrak{b})\cap\Lambda : y_{I^c}=a_{I^c}\,\}.$$

Fix a point $a_I+a_{I^c}$ in this fiber. Then $y=a_{I^c}+y_I$ lies in the fiber if and only if $y_I-a_I\in(s_I-a_I+\mathfrak{b}_I)\cap\Lambda$. By Lemma 8(ii), the cardinality of the fiber is equal to $\mathrm{vol}_{L_I\cap\Lambda}(\mathfrak{b}_I)$. Thus we obtain

$$S(s+\mathfrak{c})(\xi)\equiv \sum_{I\in\mathcal{J}^d_{\geq d_0}}\lambda(I)S(s_{I^c}+\mathfrak{b}_{I^c})(\xi)\prod_{i\in I^c}\frac{1}{1-\mathrm{e}^{\langle\xi,v_i\rangle}}\mathrm{e}^{\langle\xi,s_I\rangle}\,\mathrm{vol}_{L_I\cap\Lambda}(\mathfrak{b}_I)\prod_{i\in I}\frac{-1}{\langle\xi,v_i\rangle}$$
$$\text{mod terms of }\xi\text{-degree }\geq -d_0+1. \tag{18}$$

By Proposition 23 and Lemmas 7 and 9, the term corresponding to an $I\in\mathcal{J}^d_{\geq d_0}$ in the right-hand side of (18) is precisely

$$S^{L_I}(s+\mathfrak{c},\Lambda)(\xi)=S(s_{I^c}+\mathfrak{c}_{I^c},\Lambda_{I^c})(\xi)\,I(s_I+\mathfrak{c}_I,L_I\cap\Lambda)(\xi),$$

which completes the proof. □

Remark 25. *For $d_0=0$, we obtain the poset $\mathcal{J}^d_{\geq 0}$ of all subsets of $\{1,\dots,d\}$. The unique patching function on $\mathcal{J}^d_{\geq 0}$ is given by $\lambda(\emptyset)=1$ and $\lambda(I)=0$ for all $I\neq\emptyset$. Then the approximation is trivial, i.e., $S(s+\mathfrak{c},\Lambda)(\xi)=A^\lambda(s+\mathfrak{c},\Lambda)(\xi)$.*

Example 26. Let $\mathfrak{c}$ be the standard cone in $\mathbb{R}^2$, and $d_0=1$. Thus $\mathcal{J}^2_{\geq 1}$ consists of three subsets, $\{1\},\{2\}$ and $\{1,2\}$. A patching function is given by $\lambda(\{i\})=1$ and $\lambda(\{1,2\})=-1$. We consider the affine cone $s+\mathfrak{c}$ with $s=(-\frac12,-\frac12)$. Let $\xi=(\xi_1,\xi_2)$. We have

$$I(s_i+\mathfrak{c}_{\{i\}})(\xi)=\frac{-\mathrm{e}^{-\xi_i/2}}{\xi_i},\qquad I(s+\mathfrak{c})(\xi)=\frac{\mathrm{e}^{-\xi_1/2-\xi_2/2}}{\xi_1\xi_2},$$
$$S(s_i+\mathfrak{c}_{\{i\}})(\xi)=\frac{1}{1-\mathrm{e}^{\xi_i}},\qquad S(s+\mathfrak{c})(\xi)=\frac{1}{(1-\mathrm{e}^{\xi_1})(1-\mathrm{e}^{\xi_2})}.$$

The approximation theorem claims that

$$\frac{1}{(1-\mathrm{e}^{\xi_1})(1-\mathrm{e}^{\xi_2})}\equiv\frac{1}{1-\mathrm{e}^{\xi_2}}\cdot\frac{-\mathrm{e}^{-\xi_1/2}}{\xi_1}+\frac{1}{1-\mathrm{e}^{\xi_1}}\cdot\frac{-\mathrm{e}^{-\xi_2/2}}{\xi_2}-\frac{\mathrm{e}^{-\xi_1/2-\xi_2/2}}{\xi_1\xi_2}$$
$$\text{mod terms of }\xi\text{-degree }\geq 0.$$

Indeed, the difference between the two sides is equal to

$$\Big(\frac{1}{1-\mathrm{e}^{\xi_1}}+\frac{\mathrm{e}^{-\xi_1/2}}{\xi_1}\Big)\Big(\frac{1}{1-\mathrm{e}^{\xi_2}}+\frac{\mathrm{e}^{-\xi_2/2}}{\xi_2}\Big)$$

which is analytic near 0.

4.4. **An explicit patching function.** Next we compute an explicit patching function on $\mathcal{J}^d_{\geq d_0}$. It is related to the Möbius function of the poset $\mathcal{J}^d_{\geq d_0}$, so we call it the *Möbius patching function* and denote it by $\lambda_{\text{Möbius}}$. We will denote the corresponding patched generating function $A^{\lambda_{\text{Möbius}}}(s+\mathfrak{c},\Lambda)$ by $A_{\geq d_0}(s+\mathfrak{c},\Lambda)$.

Proposition 27. *For $I \in \mathcal{J}^d_{\geq d_0}$, let*

$$\lambda_{\text{Möbius}}(I) = (-1)^{|I|-d_0}\binom{|I|-1}{d_0-1}.$$

Then $\lambda_{\text{Möbius}}$ is a patching function on $\mathcal{J}^d_{\geq d_0}$.

Proof. We prove that $\lambda_{\text{Möbius}}$ satisfies Condition (iv) of Lemma 21. The trick is to write $\mathrm{e}^z = 1 + t(\mathrm{e}^z - 1)|_{t=1}$. Thus

$$\mathrm{e}^{z_1+\cdots+z_d} = \prod_1^d \mathrm{e}^{z_i} = \prod_{i=1}^d \big(1 + t(\mathrm{e}^{z_i}-1)\big)\Big|_{t=1}$$

Let us consider $P(t) := \prod_{i=1}^d \big(1+t(\mathrm{e}^{z_i}-1)\big) = \sum_{q=0}^d C_q(z)t^q$ as a polynomial in the indeterminate t. As $\mathrm{e}^{z_i}-1$ is a sum of terms of z_i-degree >0, we have

$$\mathrm{e}^{z_1+\cdots+z_d} \equiv \sum_{q=0}^{k_0} C_q(z) \text{ mod terms of } z\text{-degree } \geq k_0+1. \tag{19}$$

Next, we write

$$P(t) = \prod_1^d \big(1+t(\mathrm{e}^{z_i}-1)\big) = \prod_1^d \big((1-t)+t\mathrm{e}^{z_i}\big).$$

By expanding the product, we obtain

$$C_q(z) = \sum_{|K|\leq q} (-1)^{q-|K|}\binom{d-|K|}{q-|K|}\mathrm{e}^{z_K}.$$

Summing up these coefficients for $0 \leq q \leq k_0 = d-d_0$, we obtain

$$\sum_{q=0}^{k_0} C_q(z) = \sum_{|K|\leq k_0}\left(\sum_{q=|K|}^{k_0}(-1)^{q-|K|}\binom{d-|K|}{q-|K|}\right)\mathrm{e}^{z_K}.$$

By substituting $K = I^c$ and $d-q = m$, we obtain

$$\sum_{q=0}^{k_0} C_q(z) = \sum_{|I|\geq d_0} f(|I|)\mathrm{e}^{z_{I^c}},$$

with

$$f(j) = \sum_{m=d_0}^{j}(-1)^{j-m}\binom{j}{j-m} = \sum_{m=d_0}^{j}(-1)^{j-m}\binom{j}{m}.$$

The truncated binomial sum $f(j)$ is easy to compute, using the recursion relation $\binom{j}{m} = \binom{j-1}{m-1} + \binom{j-1}{m}$. We obtain

$$f(j) = (-1)^{j-d_0}\binom{j-1}{d_0-1}.$$

Thus, $\lambda_{\text{Möbius}}(I) = f(|I|)$ satisfies Condition (ii) for a patching function. □

Remark 28. *Proposition 27 can also be deduced from results in* [16].

5. Computation of the patched generating function

In this section, we show that if $k_0 = d - d_0$ is fixed, the patched generating function $A_{\geq d_0}(s + \mathfrak{c}, \Lambda)$ can be efficiently computed for a simplicial cone $s + \mathfrak{c}$. This will be a consequence of Barvinok's polynomial-time decomposition of cones in fixed dimension [9, 11]. We exhibit the dependence of the patched generating function on the vertex s explicitly as a "step function" in two useful ways, using the "ceiling" function $\lceil\cdot\rceil$ and the "fractional part" function $\{\cdot\}$, respectively.

We will write the patched generating function using the following analytic function.

Definition 29. *Let*

$$T(\tau, x) = \mathrm{e}^{\tau x}\frac{x}{1-\mathrm{e}^{x}} = -\sum_{n=0}^{\infty} B_n(\tau)\frac{x^n}{n!}, \tag{20}$$

where $B_n(\tau)$ are the Bernoulli polynomials.

We start with the following result.

Theorem 30 (Short formula for $S^{L_I}(s + \mathfrak{c}, \mathbb{Z}^d)(\xi)$ for varying s)**.** *Fix a non-negative integer k_0. There exists a polynomial time algorithm for the following problem. Given the following input:*

- (I_1) *a number d in unary encoding,*
- (I_2) *a simplicial cone $\mathfrak{c} = \mathfrak{c}(v_1, \dots, v_d) \subset \mathbb{R}^d$, represented by the primitive vectors $v_1, \dots, v_d \in \mathbb{Z}^d$ in binary encoding,*
- (I_3) *a subspace $L_I = \operatorname{lin}(v_i : i \in I) \subseteq \mathbb{R}^d$ of codimension k_0, represented by an index set $I \subseteq \{1, \dots, d\}$ of cardinality $d_0 = d - k_0$,*

compute the following output in binary encoding:

- (O_1) *a finite set Γ,*
- (O_2) *for every γ in Γ, integers $\alpha^{(\gamma)}$, rational vectors $\eta_i^{(\gamma)}$ and $w_i^{(\gamma)}$ for $i = 1, \dots, d$, where $\eta_i^{(\gamma)} \in \mathbb{Z}^d$ for $i \in I^c$*

such that for every $s \in \mathbb{Q}^d$, we have the following equality of meromorphic functions of ξ:

$$\begin{aligned} S^{L_I}&(s+\mathfrak{c},\mathbb{Z}^d)(\xi) \\ &= \sum_{\gamma\in\Gamma} \alpha^{(\gamma)} \prod_{i\in I^c} T\big(\lceil\langle \eta_i^{(\gamma)}, s\rangle\rceil, \langle \xi, w_i^{(\gamma)}\rangle\big) \\ &\qquad\qquad \cdot \prod_{i\in I} \exp\big(\langle \eta_i^{(\gamma)}, s\rangle \langle \xi, w_i^{(\gamma)}\rangle\big) \cdot \frac{1}{\prod_{i=1}^d \langle \xi, w_i^{(\gamma)}\rangle} \qquad (21a) \\ &= \mathrm{e}^{\langle \xi, s\rangle} \sum_{\gamma\in\Gamma} \alpha^{(\gamma)} \prod_{i\in I^c} T\big(\{-\langle \eta_i^{(\gamma)}, s\rangle\}, \langle \xi, w_i^{(\gamma)}\rangle\big) \cdot \frac{1}{\prod_{i=1}^d \langle \xi, w_i^{(\gamma)}\rangle}. \qquad (21b) \end{aligned}$$

Of course, for $I = \emptyset$ we have $L = \{0\}$, and so we recover formulas for $S(s+\mathfrak{c},\mathbb{Z}^d)(\xi)$. If we set $I = \{1,\dots,d\}$, then $L = \mathbb{R}^d$, and we get formulas for $I(s+\mathfrak{c},\mathbb{Z}^d)(\xi)$.

Remark 31. *Consider the term corresponding to $\gamma \in \Gamma$ in* (21a) *or* (21b)*. As it will follow from the proof, the vector $w_i^{(\gamma)}$ for $i \in I$ is just the original vector v_i, and the collection $w_i^{(\gamma)}$, $i = 1,\dots,d$, forms a basis of $\mathbb{R}^d$. Furthermore the vectors $w_i^{(\gamma)}$, with $i \in I^c$, are in L_{I^c} and form a basis of the projected lattice. The vectors $\eta_i^{(\gamma)}$, $i = 1,\dots,d$, are the dual (biorthogonal) vectors to the elements $w_j^{(\gamma)}$, $j = 1,\dots,d$, i.e., $\langle \eta_i^{(\gamma)}, w_j^{(\gamma)}\rangle = \delta_{i,j}$. Thus we only need to compute the integers $\alpha^{(\gamma)}$ and the elements $w_i^{(\gamma)}$ where $i \in I^c$.*

Remark 32. *Consider the term corresponding to $\gamma \in \Gamma$ in* (21b)*. As the vectors $w_i^{(\gamma)}$, $i \in I^c$, form a basis of the projected lattice, we may identify $V/(L_I + \Lambda)$ to $\bigoplus_{i\in I^c}[0,1[\, w_i^{(\gamma)}$. Define*

$$s^{(\gamma)} = \sum_{i\in I^c} \big\{-\langle \eta_i^{(\gamma)}, s\rangle\big\}\, w_i^{(\gamma)}.$$

As the $\eta_i^{(\gamma)}$ for $i \in I^c$ are integer vectors, and $\langle \eta_i^{(\gamma)}, v_j\rangle = 0$ if $j \in I$, we can think of $s \mapsto s^{(\gamma)}$ as a linear map on the torus $V/(\Lambda + L_I)$ with integer coefficients. The point $s + s^{(\gamma)}$ is in $\bigoplus_{i\in I^c} \mathbb{Z} w_i^{(\gamma)} \oplus \bigoplus_{i\in I} \mathbb{R} v_i$, and formula (21b) *reads also*

$$S^{L_I}(s+\mathfrak{c},\mathbb{Z}^d)(\xi) = \sum_{\gamma\in\Gamma} \alpha^{(\gamma)} \mathrm{e}^{\langle \xi, s+s^{(\gamma)}\rangle} \frac{1}{\prod_{i\in I^c}\big(1-\mathrm{e}^{\langle \xi, w_i^{(\gamma)}\rangle}\big)} \frac{1}{\prod_{i\in I}\langle \xi, v_i\rangle}. \qquad (22)$$

Now we prove the theorem.

Proof of Theorem 30. Let us describe the algorithm along the proof. Let $\Lambda = \mathbb{Z}^d$. By Proposition 23,

$$S^{L_I}(s+\mathfrak{c},\Lambda)(\xi) = S(s_{I^c} + \mathfrak{c}_{I^c}, \Lambda_{I^c})(\xi)\, I(s_I + \mathfrak{c}_I, L_I \cap \Lambda)(\xi). \qquad (23)$$

We first discuss $I(s_I + \mathfrak{c}_I, L_I \cap \Lambda)$. We have

$$I(s_I + \mathfrak{c}_I, L_I \cap \Lambda)(\xi) = \mathrm{e}^{\langle \xi, s_I \rangle} \operatorname{vol}_{L_I \cap \Lambda}(\mathfrak{b}_I) \prod_{j \in I} \frac{-1}{\langle \xi, v_j \rangle}. \tag{24}$$

Using linear functionals $\eta_i \in \mathbb{Q}^d$, $i \in I$ (the coordinate functions with respect to the basis v_i), write $s_I = \sum_{i \in I} \langle \eta_i, s \rangle v_i$. The η_i can be read off in polynomial time from the inverse of the matrix whose columns are $v_1, \dots, v_d$. Then $\mathrm{e}^{\langle \xi, s_I \rangle}$ takes the form

$$\mathrm{e}^{\langle \xi, s_I \rangle} = \prod_{i \in I} \exp(\langle \eta_i, s \rangle \langle \xi, v_i \rangle). \tag{25}$$

Now, to handle the factor $S(s_{I^c} + \mathfrak{c}_{I^c}, \Lambda_{I^c})$, note that $\mathfrak{c}_{I^c} \subset L_{I^c}$ is a k_0-dimensional cone. By using a Hermite normal form computation, which is polynomial time [29], we can compute a linear change of variables which replaces the projected lattice Λ_{I^c} on L_{I^c} by $\mathbb{Z}^{k_0}$. Then, using Barvinok's decomposition [9], we decompose it into a family of cones which are unimodular,

$$\left[\mathfrak{c}_{I^c}\right] \equiv \sum_{m \in M} \epsilon_m \left[\mathfrak{c}_{I^c}^{(m)}\right] \quad \text{(modulo cones containing lines)}, \tag{26}$$

where $\epsilon_m \in \{\pm 1\}$. As k_0 is fixed, this decomposition can be done by a polynomial time algorithm. This step is of course crucial with respect to the efficiency of the whole algorithm.

Changing again notations, we now denote by $\mathfrak{c} = \mathfrak{c}(\{w_i\}_{i \in I^c})$ one of these unimodular cones $\mathfrak{c}_{I^c}^{(m)} \subset L_{I^c}$, with primitive generators w_i, and also write $\epsilon = \epsilon_m$. We remark that the vectors w_i, $i \in I^c$, generate the projected lattice on L_{I^c}. Using linear functionals $\eta_i \in \mathbb{Q}^d$, $i \in I^c$, write $s_{I^c} = \sum_{i \in I^c} \langle \eta_i, s \rangle w_i$. Actually, we have $\eta_i \in \mathbb{Z}^d$. By letting $w_i = v_i$ for the other indices $i \in I$, we can write

$$s = \sum_{i \in I} \langle \eta_i, s \rangle v_i + \sum_{i \in I^c} \langle \eta_i, s \rangle w_i = \sum_{i=1}^{d} \langle \eta_i, s \rangle w_i. \tag{27}$$

Let s'_{I^c} be the unique lattice point in the fundamental parallelepiped of the cone $s_{I^c} + \mathfrak{c}$. We have

$$s'_{I^c} = \sum_{i \in I^c} \lceil \langle \eta_i, s \rangle \rceil w_i. \tag{28}$$

Using this, we obtain the generating function from Lemma 9 as

$$S(s_{I^c} + \mathfrak{c})(\xi) = \frac{\mathrm{e}^{\langle \xi, s'_{I^c} \rangle}}{\prod_{i \in I^c} (1 - \mathrm{e}^{\langle \xi, w_i \rangle})}.$$

Thus finally, using (24) we have the meromorphic function

$$\epsilon \operatorname{vol}_{L_I \cap \Lambda}(\mathfrak{b}_I) (-1)^{|I|} \cdot \frac{\mathrm{e}^{\langle \xi, s'_{I^c} \rangle}}{\prod_{i \in I^c} (1 - \mathrm{e}^{\langle \xi, w_i \rangle})} \cdot \frac{\mathrm{e}^{\langle \xi, s_I \rangle}}{\prod_{j \in I} \langle \xi, v_j \rangle}. \tag{29}$$

Then (29) is now written

$$\alpha \prod_{i\in I^c} T(\lceil\langle\eta_i, s\rangle\rceil, \langle\xi, w_i\rangle) \cdot \prod_{i\in I} \exp(\langle\eta_i, s\rangle\langle\xi, v_i\rangle) \cdot \frac{1}{\prod_{i=1}^d \langle\xi, w_i\rangle}, \tag{30}$$

where α collects the multiplicative constants in (29). Collecting these terms gives the desired short formula (21a).

To derive the second form, we note $\lceil\langle\eta_i, s\rangle\rceil = \langle\eta_i, s\rangle + \{-\langle\eta_i, s\rangle\}$, so we can write

$$T(\lceil\langle\eta_i, s\rangle\rceil, \langle\xi, w_i\rangle) = T(\{-\langle\eta_i, s\rangle\}, \langle\xi, w_i\rangle) \exp(\langle\eta_i, s\rangle\langle\xi, w_i\rangle). \tag{31}$$

Thus the term (30) can be written as

$$\alpha \prod_{i\in I^c} T(\{-\langle\eta_i, s\rangle\}, \langle\xi, w_i\rangle) \cdot \mathrm{e}^{\langle\xi, s\rangle} \cdot \frac{1}{\prod_{i=1}^d \langle\xi, w_i\rangle}, \tag{32}$$

using (27). Collecting these terms gives the short formula (21b). □

Example 33. Let us give an example of the output of our algorithm, in a small example. Consider the 3-dimensional cone with rays given by the vectors $(1, 1, 1)$, $(1, -1, 0)$, $(1, 1, 0)$. This cone is not unimodular. We consider the affine cone $s+\mathfrak{c}$. Our algorithm described in Theorem 30 computes any intermediate generating function $S^L(s+\mathfrak{c}, \mathbb{Z}^3)$ when L is a linear span of a face of $\mathfrak{c}$. For $L = \{0\}$ (indexed by the empty set I), we obtain the meromorphic function $S(s+\mathfrak{c}, \mathbb{Z}^3)(\xi)$ (the discrete generating function of the cone $s+\mathfrak{c}$). Here $S(s+\mathfrak{c}, \mathbb{Z}^3)(\xi)$ depends of $s = (s_1, s_2, s_3)$ and $\xi = (\xi_1, \xi_2, \xi_3)$ and is given by:

$$\begin{aligned}&\exp\left(s_1\xi_1 + s_2\xi_2 + s_3\xi_3\right)\\ &\quad\Bigg(-\frac{T\left(\{-s_3 + s_2\}, -\xi_1 - \xi_2\right) T\left(\{-s_1 + s_2\}, \xi_1\right) T\left(\{-s_3\}, \xi_1 + \xi_2 + \xi_3\right)}{\left(-\xi_1 - \xi_2\right) \xi_1 \left(\xi_1 + \xi_2 + \xi_3\right)}\\ &\qquad + \frac{T\left(\{-s_3 + s_2\}, \xi_1 - \xi_2\right) T\left(\{2\, s_3 - s_2 - s_1\}, \xi_1\right) T\left(\{-s_3\}, \xi_1 + \xi_2 + \xi_3\right)}{\left(\xi_1 - \xi_2\right) \xi_1 \left(\xi_1 + \xi_2 + \xi_3\right)}\Bigg).\end{aligned}$$

If $L = \mathbb{R}v_1$ is the subspace of dimension 1 generated by the edge $v_1 = (1, 1, 1)$ of the cone $\mathfrak{c}$ (so that L is indexed by the subset $I = \{1\}$ of $\{1, 2, 3\}$), the intermediate generating function $S^L(s + \mathfrak{c}, \mathbb{Z}^3)$ is given by:

$$\begin{aligned}\exp\left(s_1\xi_1 + s_2\xi_2 + s_3\xi_3\right) &\Bigg(-\frac{T\left(\{-s_3 + s_2\}, -\xi_1 - \xi_2\right) T\left(\{-s_1 + s_2\}, \xi_1\right)}{\left(-\xi_1 - \xi_2\right) \xi_1 \left(-\xi_1 - \xi_2 - \xi_3\right)}\\ &\quad + \frac{T\left(\{-s_3 + s_2\}, \xi_1 - \xi_2\right) T\left(\{2\, s_3 - s_2 - s_1\}, \xi_1\right)}{\left(\xi_1 - \xi_2\right) \xi_1 \left(-\xi_1 - \xi_2 - \xi_3\right)}\Bigg).\end{aligned}$$

Remark 34. *When $k_0 = d - d_0$ is fixed, the set $\mathcal{J}^d_{\geq d_0}$ has a polynomially bounded cardinality, and it can be enumerated by a straightforward algorithm along with the evaluation of the patching function $\lambda_{\text{Möbius}}$. Thus we can also compute $A_{\geq d_0}(s + \mathfrak{c}, \Lambda)(\xi)$ in the same form* (21a) *or* (21b) *in polynomial time.*

6. Computation of Ehrhart quasi-polynomials

We now apply the approximation of the generating functions of the cones at vertices to the computation of the highest coefficients for a weighted Ehrhart quasi-polynomial. We first discuss the case when the weight is a power of a linear form.

Theorem 35. *Let $\mathfrak{p}$ be a simple polytope and let $\mathcal{V}(\mathfrak{p})$ denote the set of its vertices. For each vertex $s \in \mathcal{V}(\mathfrak{p})$, let $\mathfrak{c}_s$ be the tangent cone of s, and let $q_s \in \mathbb{N}$ be a positive integer such that $q_s\, s \in \Lambda$. Fix a linear form $\ell \in V^*$ and M a nonnegative integer. Fix $0 \le k_0 \le d$ and let $d_0 = \max\{d - k_0, 0\}$. Then the Ehrhart quasi-polynomial*

$$E(\mathfrak{p}, \ell, M; n) = \sum_{x \in n\mathfrak{p} \cap \Lambda} \frac{\langle \ell, x\rangle^M}{M!}$$

coincides in degree $\ge M + d - k_0$ with the following quasi-polynomial

$$\sum_{k=0}^{k_0} \sum_{s \in \mathcal{V}(\mathfrak{p})} (\lfloor n \rfloor_{q_s})^{M+d-k} \frac{\langle \xi, s\rangle^{M+d-k}}{(M+d-k)!} A_{\ge d_0}\big(\{n\}_{q_s} s + \mathfrak{c}_s, \Lambda\big)_{[-d+k]}(\xi), \tag{33a}$$

evaluated at $\xi = \ell$, which can also be written as

$$\sum_{k=0}^{k_0} n^{M+d-k} \sum_{s \in \mathcal{V}(\mathfrak{p})} \frac{\langle \xi, s\rangle^{M+d-k}}{(M+d-k)!} \Big(\mathrm{e}^{-\langle \xi, \{n\}_{q_s} s\rangle} A_{\ge d_0}\big(\{n\}_{q_s} s + \mathfrak{c}_s, \Lambda\big)(\xi)\Big)_{[-d+k]}, \tag{33b}$$

evaluated at $\xi = \ell$.

In the following, we will use the second form (33b).

Remark 36. *The sum (33) depends polynomially on ℓ. However, for an individual vertex s, the functions*

$$\xi \mapsto \frac{\langle \xi, s\rangle^{M+d-k}}{(M+d-k)!} A_{\ge d_0}\big(\{n\}_{q_s} s + \mathfrak{c}_s, \Lambda\big)_{[-d+k]}(\xi)$$

and

$$\xi \mapsto \frac{\langle \xi, s\rangle^{M+d-k}}{(M+d-k)!} \Big(\mathrm{e}^{-\langle \xi, \{n\}_{q_s} s\rangle} A_{\ge d_0}\big(\{n\}_{q_s} s + \mathfrak{c}_s, \Lambda\big)(\xi)\Big)_{[-d+k]}$$

are meromorphic functions, which are not defined if ξ is singular. Thus in the algorithm we use a deformation procedure.

Proof of Theorem 35. The sum $\sum_{x \in n\mathfrak{p} \cap \Lambda} \frac{\langle \xi, x\rangle^M}{M!}$ is the term of ξ-degree M in

$$S(n\mathfrak{p})(\xi) = \sum_{s \in \mathcal{V}(\mathfrak{p})} S(ns + \mathfrak{c}_s)(\xi).$$

Fix a vertex s. We write $n = \lfloor n \rfloor_{q_s} + \{n\}_{q_s}$. As $\lfloor n \rfloor_{q_s} s$ is a lattice point, we have

$$S(ns + \mathfrak{c}_s)(\xi) = \mathrm{e}^{\lfloor n \rfloor_{q_s} \langle \xi, s\rangle} S(\{n\}_{q_s} s + \mathfrak{c}_s)(\xi). \tag{34}$$

Consider $S(ns+\mathfrak{c}_s)_{[M]}(\xi)$ as a quasi-polynomial in n. By (34), it coincides in degree $\geq M+d-k_0$ with

$$\sum_{k=0}^{k_0}(\lfloor n\rfloor_{q_s})^{M+d-k}\frac{\langle\xi,s\rangle^{M+d-k}}{(M+d-k)!}\,S(\{n\}_{q_s}s+\mathfrak{c}_s)_{[-d+k]}(\xi).$$

Now, for $0\leq k\leq k_0$, we have

$$S(\{n\}_{q_s}s+\mathfrak{c}_s)_{[-d+k]}(\xi)=A_{\geq d_0}(\{n\}_{q_s}s+\mathfrak{c}_s)_{[-d+k]}(\xi).$$

By specializing on $\xi=\ell$, we obtain the claim in the form of equation (33a).

To obtain the second claim in the form of (33b), we write

$$S(ns+\mathfrak{c}_s)(\xi)=\mathrm{e}^{n\langle\xi,s\rangle}\bigl(\mathrm{e}^{-\langle\xi,s\rangle\{n\}_{q_s}}S(\{n\}_{q_s}s+\mathfrak{c}_s)(\xi)\bigr).\tag{35}$$

Again, by expanding we obtain that the quasi-polynomial $S(ns+\mathfrak{c}_s)_{[M]}(\xi)$ coincides in degree $\geq M+d-k_0$ with

$$\sum_{k=0}^{k_0}n^{M+d-k}\frac{\langle\xi,s\rangle^{M+d-k}}{(M+d-k)!}\Bigl(\mathrm{e}^{-\langle\xi,s\rangle\{n\}_{q_s}}S(\{n\}_{q_s}s+\mathfrak{c}_s)(\xi)\Bigr)_{[-d+k]}.$$

Since $\mathrm{e}^{-\langle\xi,s\rangle\{n\}_{q_s}}$ is analytic in ξ, we have for $0\leq k\leq k_0$ that

$$\begin{aligned}&\Bigl(\mathrm{e}^{-\langle\xi,s\rangle\{n\}_{q_s}}S(\{n\}_{q_s}s+\mathfrak{c}_s)(\xi)\Bigr)_{[-d+k]}\\&\qquad=\Bigl(\mathrm{e}^{-\langle\xi,s\rangle\{n\}_{q_s}}A_{\geq d_0}(\{n\}_{q_s}s+\mathfrak{c}_s)(\xi)\Bigr)_{[-d+k]}.\end{aligned}\tag{36}$$

Again, by specializing on $\xi=\ell$, we obtain the claim in the form of equation (33b). □

We now derive the coefficients of the weighted Ehrhart polynomial as short *closed formulas* that are “step polynomials” (cf. [34]). These can then be evaluated efficiently, providing a corollary (Theorem 42) in the same form as Barvinok's theorem in [10].

Theorem 37. *For every fixed number $k_0\in\mathbb{N}$, there exists a polynomial-time algorithm for the following problem.*
Input:

- (I_1) *a number $d\in\mathbb{N}$ in unary encoding, with $d\geq k_0$,*
- (I_2) *a finite index set $\mathcal{V}$,*
- (I_3) *a simple polytope $\mathfrak{p}$, given by its vertices, rational vectors $s_j\in\mathbb{Q}^d$ for $j\in\mathcal{V}$ in binary encoding,*
- (I_4) *a rational vector $\ell\in\mathbb{Q}^d$ in binary encoding,*
- (I_5) *a number $M\in\mathbb{N}$ in unary encoding.*

Output, in binary encoding,

- (O_1) *an index set Γ,*
- (O_2) *polynomials $f^{\gamma,m}\in\mathbb{Q}[r_1,\dots,r_{k_0}]$ and integer numbers $\zeta_i^{\gamma,m}\in\mathbb{Z}$, $q_i^{\gamma,m}\in\mathbb{N}$ for $\gamma\in\Gamma$ and $m=M+d-k_0,\dots,M+d$ and $i=1,\dots,k_0$,*

such that the Ehrhart quasi-polynomial

$$E(\mathfrak{p}, \ell, M; n) = \sum_{x \in n\mathfrak{p} \cap \Lambda} \frac{\langle \ell, x \rangle^M}{M!} = \sum_{m=0}^{M+d} E_m(\mathfrak{p}, \ell, M; \{n\}_q)\, n^m$$

agrees in n-degree $\geq M + d - k_0$ with the quasi-polynomial

$$\sum_{\gamma \in \Gamma} \sum_{m=M+d-k_0}^{M+d} f^{\gamma,m}\Big(\{\zeta_1^{\gamma,m} n\}_{q_1^{\gamma,m}}, \dots, \{\zeta_{k_0}^{\gamma,m} n\}_{q_{k_0}^{\gamma,m}}\Big)\, n^m.$$

Remark 38. *For $d \leq k_0$, the algorithm actually computes the complete Ehrhart quasi-polynomial, i.e., the coefficient functions $E_m(\mathfrak{p}, \ell, M; \{n\}_q)$ for $m = 0, \dots, M+d$. The key point of our method, however, is to handle the case where $d > k_0$; then the non-trivial efficiently computable approximations come into play.*

Remark 39. *The specific form of the quasi-polynomial given by the theorem gives a more precise period q_i for the individual terms, rather than a period q_s that is determined by the vertex. The q_i will always be divisors of q_s. Due to the projections into lattices in small dimension $\leq k_0$, these periods can be much smaller than q_s. In particular, the highest-degree coefficient E_{M+d} of course is a constant.*

We will use the following lemma.

Lemma 40 (Lemma 4 of [2])**.** *For every fixed number $D \in \mathbb{N}$, there exists a polynomial time algorithm for the following problem.*
Input: a number M in unary encoding, a sequence of k polynomials $P_j \in \mathbb{Q}[X_1, \dots, X_D]$ of total degree at most M, in dense monomial representation.
Output: the product $P_1 \cdots P_k$ truncated at degree M.

We can now prove the theorem.

Proof of Theorem 37. Because the polytope $\mathfrak{p}$ is simple, we can use the primal–dual algorithm by Bremner, Fukuda, and Marzetta [17, Corollary 1], to compute the inequality description (H-description) from the given V-description in polynomial time. From the double description, we can compute in polynomial time the description of the tangent cones $\mathfrak{c}_{s_j}$ for $j \in \mathcal{V}$ by the primitive vectors $v_{s_j,1}, \dots, v_{s_j,d} \in \mathbb{Z}^d$ such that $\mathfrak{c}_{s_j} = \mathfrak{c}(v_{s_j,1}, \dots, v_{s_j,d})$.

We now use formula (33b) of Theorem 35, which gives (with $d_0 = d - k_0$)

$$E_m(p, \xi, M; \{n\}_q) = \sum_{s \in \mathcal{V}(\mathfrak{p})} \frac{\langle \xi, s \rangle^m}{m!} \Big(\mathrm{e}^{-\langle \xi, \{n\}_{q_s} s \rangle} A_{\geq d_0}\big(\{n\}_{q_s} s + \mathfrak{c}_s, \Lambda\big)(\xi) \Big)_{[-d+k]} \quad (37)$$

for $m = M+d-k$, when $m \geq M+d-k_0$. We compute this separately for each $k = 0, \dots, k_0$, that is, $m = M + d - k_0, \dots, M + d$. Let $s + \mathfrak{c}_s$ be one of these cones. By the algorithm of Theorem 30 and Remark 34, we compute the data describing the parametric short formula (21b) for $A_{\geq d_0}\big(\{n\}_{q_s}s + \mathfrak{c}_s, \Lambda\big)(\xi)$. We then consider one of the summands of

$$\frac{\langle \xi, s\rangle^m}{m!}\Big(\mathrm{e}^{-\langle \xi, \{n\}_{q_s}s\rangle} A_{\geq d_0}\big(\{n\}_{q_s}s + \mathfrak{c}_s, \Lambda\big)(\xi)\Big)_{[-d+k]}$$

at a time. Here $\mathrm{e}^{-\langle \xi, \{n\}_{q_s}s\rangle}$ and the term $\mathrm{e}^{\langle \xi, \{n\}_{q_s}s\rangle}$ from (21b) cancel, and thus each summand takes the form

$$\Big(\frac{\langle \xi, s\rangle^{M+d-k}}{(M+d-k)!}\Big)\Big(\prod_{i\in I^c} T\big(\tau_i(n), \langle \xi, w_i\rangle\big)\Big)_{[k]}\Big(\frac{1}{\prod_{i=1}^d \langle \xi, w_i\rangle}\Big) \tag{38}$$

where

$$\tau_i(n) := \big\{-\langle \eta_i, s\rangle \{n\}_{q_s}\big\} \quad \text{for } i \in I^c. \tag{39}$$

Let $q_i \in \mathbb{N}$ be the smallest positive integer such that $q_i\langle -\eta_i, s\rangle \in \mathbb{Z}$. Then q_i is a divisor of the number q_s associated with the vertex s, because $\eta_i \in \mathbb{Z}^d$. Then

$$\tau_i(n) = \tfrac{1}{q_i}\big\{\zeta_i\{n\}_{q_s}\big\}_{q_i} \quad \text{with} \quad \zeta_i = q_i\langle -\eta_i, s\rangle \in \mathbb{Z}.$$

Since q_i is a divisor of q_s, this simplifies to

$$\tau_i(n) = \tfrac{1}{q_i}\big\{\zeta_i n\big\}_{q_i}, \tag{40}$$

where of course ζ_i can be reduced modulo q_i as well because n is assumed to be an integer. We now treat $r_i := \{\zeta_i n\}_{q_i} \in \mathbb{N}$ as symbolic variables.

In order to evaluate (38) at $\xi = \ell$, we note the following. The first factor is holomorphic in ξ and homogeneous of ξ-degree $m = M + d - k$, the second factor is holomorphic in ξ and homogeneous of degree k, and the third factor is homogeneous of ξ-degree $-d$. If $\langle \ell, w_i\rangle = 0$ for some i, we cannot just substitute $\xi = \ell$ in the formula. Instead we use a perturbation. In polynomial time, we can compute a rational vector $\ell' \in \mathbb{Q}^d$ such that $\langle \ell', w_i\rangle \neq 0$ for all vectors w_i with $\langle \ell, w_i\rangle = 0$. It is important that we choose the same vector once and for all computations with all cones and summands.

We then set $\xi = t(\ell + \epsilon\ell')$, where t and ϵ are treated as symbolic variables. Here the exponent of the variable t keeps track of the ξ-grading. We then do computations with truncated series in $\mathbb{Q}[r_i : i \in I^c][t^{\pm 1}, \epsilon^{\pm 1}]$. We note that this is a polynomial ring in a constant number of variables only, because $|I^c|$ is bounded above by the constant k_0. Thus Lemma 40 gives us a polynomial-time algorithm for multiplying the series. Then (38) can be written as:

$$\frac{\langle \ell + \epsilon\ell', s\rangle^m}{m!} \cdot \Big(\prod_{i\in I^c} T\big(\tau_i(n), \langle t(\ell + \epsilon\ell'), w_i\rangle\big)\Big)_{[k]} \cdot \frac{1}{\prod_{i=1}^d \langle \ell + \epsilon\ell', w_i\rangle} \cdot t^{M-k}, \tag{41}$$

where the subscript $[k]$ now means to take the term of t-degree k. In the end we are interested in the coefficient of the term $t^M\epsilon^0$.

Expanding the factors of (41) gives the following contributions, all of which can be written down in polynomial time. First of all, the rational terms $\langle \ell + \epsilon\ell', w_i\rangle^{-1}$ give the following contribution. If $\langle \ell, w_i\rangle = 0$, we simply get

$$\frac{1}{\langle \ell + \epsilon\ell', w_i\rangle} = \frac{1}{\langle \ell', w_i\rangle}\epsilon^{-1}. \tag{42}$$

If $\langle \ell, w_i\rangle \neq 0$, we get the geometric series in ϵ

$$\frac{1}{\langle \ell + \epsilon\ell', w_i\rangle} = \frac{1}{\langle \ell, w_i\rangle}\sum_{u=0}^{\infty}\left(-\frac{\langle \ell', w_i\rangle}{\langle \ell, w_i\rangle}\right)^u \epsilon^u.$$

The first and second terms in (41) are holomorphic, thus the only negative degrees in ϵ come from the rational terms (42). Let U be the number of vectors w_i that are orthogonal to ℓ; then ϵ^{-U} is the lowest negative degree. Note that $U \leq d$. Since we wish to find the term of ϵ-degree 0, we can truncate all series after ϵ-degree U:

$$\frac{1}{\langle \ell + \epsilon\ell', w_i\rangle} = \frac{1}{\langle \ell, w_i\rangle}\sum_{u=0}^{U}\left(-\frac{\langle \ell', w_i\rangle}{\langle \ell, w_i\rangle}\right)^u \epsilon^u + o_\epsilon(\epsilon^U). \tag{43}$$

We expand the first factor of (41) as follows.

$$\frac{\langle \ell + \epsilon\ell', s\rangle^m}{m!} = \sum_{u=0}^{\min\{m,U\}}\binom{m}{u}\langle \ell, s\rangle^{m-u}\langle \ell', s\rangle^u\epsilon^u + o_\epsilon(\epsilon^U). \tag{44}$$

Now we consider the holomorphic terms

$$\begin{aligned}
&T(\tau_i(n), \langle t(\ell + \epsilon\ell'), w_i\rangle)\\
&\quad= -\sum_{j=0}^{\infty}\frac{1}{j!}B_j(\tau_i)\left\langle t(\ell + \epsilon\ell'), w_i\right\rangle^j\\
&\quad= -\sum_{j=0}^{k_0}\frac{1}{j!}B_j\left(\tfrac{1}{q_i}r_i\right)\left(\sum_{u=0}^{\min\{j,U\}}\binom{j}{u}\langle \ell, w_i\rangle^{j-u}\langle \ell', w_i\rangle^u\epsilon^u\right)t^j\\
&\qquad\qquad + o_t(t^{k_0}) + o_\epsilon(\epsilon^U).
\end{aligned} \tag{45}$$

The Bernoulli polynomials $B_j(\tau_i)$ of degree $j \leq k_0$ that appear in this formula can be efficiently expanded in polynomial time using recursion formulas. We remark that the variables r_i appear with a degree that is at most that of t. Using Lemma 40, we multiply the truncated series (45) for $i \in I^c$ in $\mathbb{Q}[r_i : i \in I^c][t][\epsilon]$, truncating in each step after t^{k_0} and ϵ^U. We thus obtain the second factor of (41),

$$\Bigl(\prod_{i\in I^c} T\bigl(\tau_i(n), \bigl\langle t(\ell + \epsilon\ell'), w_i\bigr\rangle\bigr)\Bigr)_{[k]} \quad \text{for all } k = 0, \dots, k_0, \tag{46}$$

as a truncated series in $\mathbb{Q}[r_i : i \in I^c][\epsilon]$.

Then we multiply the truncated series (42), (43), (46), and (44) in polynomial time, truncating in each step after ϵ^U, using Lemma 40. In the end, we read out the coefficient of ϵ^0 as a polynomial in $\mathbb{Q}[r_i : i \in I^c]$. Then we substitute for r_i. Collecting these terms gives the formula for the Ehrhart coefficient $E_m(\mathfrak{p},\xi,M;\{n\}_q)$. $\square$

Example 41. Let us give a small example of the output of our algorithm for $E_m(\mathfrak{p},\ell,M,\{n\}_q)$, when $\mathfrak{p}$ is the simplex in $\mathbb{R}^5$ with vertices:

$$(0,0,0,0,0),\ (\tfrac{1}{2},0,0,0,0),\ (0,\tfrac{1}{2},0,0,0),\ (0,0,\tfrac{1}{2},0,0),\ (0,0,0,\tfrac{1}{6},0),\ (0,0,0,0,\tfrac{1}{6}).$$

We consider the linear form ℓ on $\mathbb{R}^5$ given by the scalar product with $(1,1,1,1,1)$.

If $M=0$, the coefficients of $E_m(\mathfrak{p},\ell,M=0;\{n\}_q)$ are just the coefficients of the unweighted Ehrhart quasi-polynomial $S(n\mathfrak{p},1)$. We obtain

$$\begin{aligned} S(n\mathfrak{p},1) = \frac{1}{34560}n^5 + \Big(\frac{5}{3456} - \frac{1}{6912}\{n\}_2\Big)n^4 \\ + \Big(\frac{139}{5184} - \frac{5}{864}\{n\}_2 + \frac{1}{3456}(\{n\}_2)^2\Big)n^3 + \cdots . \end{aligned}$$

Now if $M=1$, all integral points (x_1,x_2,x_3,x_4,x_5) are weighted with the function $h(x)=x_1+x_2+x_3+x_4+x_5$, and we obtain

$$\begin{aligned} S(n\mathfrak{p},h) = \frac{11}{1244160}n^6 + \Big(\frac{19}{41472} - \frac{11}{207360}\{n\}_2\Big)n^5 \\ + \Big(\frac{553}{62208} - \frac{95}{41472}\{n\}_2 + \frac{11}{82944}(\{n\}_2)^2\Big)n^4 + \cdots . \end{aligned}$$

We can remark that although $q=6$ is the smallest integer such that $q\mathfrak{p}$ is a lattice polytope, only periodic functions of n mod 2 enter in the top three Ehrhart coefficients. This is indeed conform to the known periodicity properties of the Ehrhart coefficients.

As a corollary, simply by evaluating the step polynomials, we obtain the following result, which directly extends the complexity result from Barvinok's paper to the weighted case.

Theorem 42 (Evaluation of the Ehrhart coefficients for a given dilation class $\{n\}_q$)**.** *For every fixed number $k_0\in\mathbb{N}$, there exists a polynomial-time algorithm for the following problem.*
Input:

- (I$_1$) *a number $d\in\mathbb{N}$ in unary encoding, with $d\geq k_0$,*
- (I$_2$) *a finite index set $\mathcal{V}$,*
- (I$_3$) *a simple polytope $\mathfrak{p}$, given by its vertices, rational vectors $s_j\in\mathbb{Q}^d$ for $j\in\mathcal{V}$ in binary encoding,*
- (I$_4$) *a rational vector $\ell\in\mathbb{Q}^d$ in binary encoding,*
- (I$_5$) *a number $M\in\mathbb{N}$ in unary encoding,*
- (I$_6$) *a number n in binary encoding,*

Output, in binary encoding,

- (O$_1$) *a positive integer $q\in\mathbb{N}$ such that $q\mathfrak{p}$ is a lattice polytope and*
- (O$_2$) *the numbers $E_m(\mathfrak{p},\ell,M;\{n\}_q)$ for $m=M+d-k_0,\dots,M+d$.*

Remark 43. *A direct algorithm for computing $E_m(\mathfrak{p},\ell,M;\{n\}_q)$ for just one dilation class $\{n\}_q$ could of course use the values $r_i=\{\zeta_i n\}_{q_i}\in\mathbb{Z}$ rather* *than symbolic variables r_i and would therefore only need to do calculations with truncated series in the two-variable ring $\mathbb{Q}[t^{\pm 1}, \epsilon^{\pm 1}]$.*

Via the decomposition of polynomials into powers of linear forms, which is, as discussed in [2], polynomial-time under suitable hypotheses, we obtain the following corollary.

Corollary 44. *For every fixed number $k_0 \in \mathbb{N}$, there exist polynomial-time algorithms for the following problems.*
Input:

- (I_1) *a number $d \in \mathbb{N}$ in unary encoding, with $d \geq k_0$,*
- (I_2) *a simple rational polytope $\mathfrak{p} \subset \mathbb{R}^d$, given by its vertices in binary encoding,*
- (I_3) *a number M in unary encoding,*
- (I_4) *a polynomial h of degree $\leq M$ which is given either as*
  - (a) *a power of a linear form, or*
  - (b) *a sparse polynomial where each monomial only depends on a fixed number of variables, or*
  - (c) *a sparse polynomial of fixed total degree,*
- (I_5) *a number n in binary encoding,*

Output, in binary encoding,

- (O_1) *a positive integer $q \in \mathbb{N}$ such that $q\mathfrak{p}$ is a lattice polytope and*
- (O_2) *the numbers $E_m(\mathfrak{p}, h; \{n\}_q)$ for $m = M + d - k_0, \ldots, M + d$.*

7. Experiments

We implemented the algorithms in *Maple*, for the unweighted case and assuming that the input were lattice simplices of full dimension (in this case the quasi-polynomial becomes a polynomial). This assumption was made for simplicity of output in the calculation and because available software to verify the results (e.g., *LattE macchiato* [31]) cannot compute with weights. In addition, already the problem of computing Ehrhart polynomials for lattice simplices has received attention by many researchers and it is non-trivial (see e.g., the references in [21]). After checking simple low-dimensional examples by hand, we set up automatic scripts for generating random tests. The simplices generated had vertex coordinates drawn uniformly at random from $\{-99, \ldots, 99\}$. We timed the speed of the procedure to compute the top three Ehrhart coefficients in 50 random simplices per dimension and recorded the average time of computation. We compared with the computation of the *full* Ehrhart polynomials using the state-of-the-art algorithms implemented in *LattE macchiato* [31]; see Table 1.

In the table, *Dual* refers to an implementation of Barvinok's decomposition of the duals of the tangent cones into unimodular cones, as implemented first in *LattE* [25], and which is still the default method in *LattE macchiato*.[1] *Primal* refers to a primal variant of Barvinok's decomposition described

[1]The LattE macchiato command is `count --ehrhart-polynomial`.

TABLE 1. Computation times for Ehrhart polynomials of random lattice simplices

	Average runtime (CPU seconds)			
	Full (*LattE macchiato*)			Top 3
Dimension	Dual	Primal	Primal_{1000}	(new code)
3	0.16	0.10	0.04	1.12
4	28.00	4.68	0.28	4.31
5		317.5	5.8	13.4
6			198.0	37.4
7				103
8				294
9				393
10				1179
11				1681

in [30]; it is more efficient for these examples because the determinants of the dual cones are much larger.[2] We remark that our implementation of the new algorithm in Maple also uses a primal variant of Barvinok's decomposition to unimodular cones, which was introduced in [20]. Thus the new code should be compared to the runtimes listed in column *Primal*. Finally, $Primal_{1000}$ refers to a variant in which Barvinok's decomposition is stopped when a cone has a determinant at most 1000; then the points in the fundamental parallelepipeds are enumerated.[3]

All computations were stopped if unfinished after 30 minutes, thus the table ends at dimension 11 because all randomly generated examples we tried in dimension 12 took more than 30 minutes of calculation. The computation times are given in CPU seconds on a computer with AMD Opteron 880 processors running at 2.4 GHz.

In conclusion, the experiments indicate that the algorithms presented here can lead to dramatic improvements upon the computation of full Ehrhart polynomials. The fact that, for very low dimensions, the implementation is slower than *LattE macchiato*, is explained by the choice of *Maple* as an implementation language. *Maple* is an interpreted system, which is much slower than C++, the implementation language of *LattE macchiato*. We expect that the speedups of $Primal_{1000}$ compared to *Primal*, which were

[2] The command is `count --ehrhart-polynomial --irrational-primal`.

[3] The command used is `count --ehrhart-polynomial --irrational-primal --maxdet=1000`.

first documented in [30], will also be obtained in a refined implementation of our new algorithms.

The implementation is available at [3].

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Velleda Baldoni: Dipartimento di Matematica, Università degli studi di Roma "Tor Vergata", Via della ricerca scientifica 1, I-00133 Roma, Italy
E-mail address: baldoni@mat.uniroma2.it

Nicole Berline: Centre de Mathématiques Laurent Schwartz, École Polytechnique, 91128 Palaiseau Cedex, France
E-mail address: nicole.berline@math.polytechnique.fr

Jesús A. De Loera: Department of Mathematics, University of California, Davis, One Shields Avenue, Davis, CA, 95616, USA
E-mail address: deloera@math.ucdavis.edu

Matthias Köppe: Department of Mathematics, University of California, Davis, One Shields Avenue, Davis, CA, 95616, USA
E-mail address: mkoeppe@math.ucdavis.edu

Michèle Vergne: Institut de Mathématiques de Jussieu, Théorie des Groupes, Case 7012, 2 Place Jussieu, 75251 Paris Cedex 05, France
E-mail address: vergne@math.jussieu.fr