



The equivariant Todd genus of a complete toric variety, with Danilov condition

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Abstract

We write the *equivariant* Todd class of a general complete toric variety as an explicit combination of the orbit closures, the coefficients being *analytic* functions on the Lie algebra of the torus which satisfy Danilov's requirement.

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1. Introduction

In [1], to any rational affine cone \mathfrak{a} in a rational vector space with a rational scalar product, we associated in a canonical way a holomorphic function with rational Taylor coefficients $\mu(\mathfrak{a})$. These functions are defined recursively by an elementary combinatorial construction and $\mu(\mathfrak{a})$ can be computed in polynomial time at any order, when $\dim \mathfrak{a}$ is fixed.

In the present article, these μ -functions are used to write the *equivariant* Todd class of a general complete toric variety as an explicit combination of the orbit closures, the coefficients being (germs of) *analytic* functions on \mathfrak{g} which satisfy Danilov's requirement [6]. Let $G \simeq (\mathbb{C}^*)^d$

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be a complex torus. Let σ be a rational cone in the lattice of one parameter subgroups of G , and let $\check{\sigma}$ be the quotiented dual cone (see Definition 7). Then for any fan Σ in the lattice, the equivariant Todd class $\text{Todd}(X_\Sigma)$ of the corresponding G -toric variety X_Σ is written as the combination

$$\text{Todd}(X_\Sigma) = \sum_{\sigma \in \Sigma} \mu(\check{\sigma})[\bar{O}_\sigma] \tag{1}$$

where $[\bar{O}_\sigma]$ is the equivariant homology class of the G -orbit closure \bar{O}_σ . Notice the special features of this formula: all orbits closures \bar{O}_σ appear and, for each σ , the term $\mu(\check{\sigma})$ is a germ of analytic function (with rational Taylor coefficients) on \mathfrak{g} which depends only on the cone σ , not on the fan (Danilov condition).

Let us write (1) in the case of the projective line $\mathbb{P}_1(\mathbb{C})$. There are three orbit closures, $\mathbb{P}_1(\mathbb{C})$ itself corresponding to the cone $\{0\}$, and the north and south poles p_+ , p_- corresponding respectively to the cones \mathbb{R}_+ and $-\mathbb{R}_+$.

Define

$$B(\xi) = \frac{1}{e^\xi - 1} - \frac{1}{\xi} = \sum_{n \geq 1} \frac{b(n)}{n!} \xi^{n-1},$$

where $b(n)$ are the Bernoulli numbers. We obtain the following expression for the equivariant Todd class of $\mathbb{P}_1(\mathbb{C})$:

$$\text{Todd}(\mathbb{P}_1(\mathbb{C}))(\xi) = [\mathbb{P}_1(\mathbb{C})] - B(\xi)[p_+] - B(-\xi)[p_-]. \tag{2}$$

The construction of this paper extends to equivariant homology a similar result obtained by Pommersheim–Thomas [9] for the ordinary Todd class. The construction of [9] requires a choice of a complement map. In particular, a scalar product provides a natural complement map. Previously, Morelli [8] gave another type of construction for the ordinary Todd class, satisfying Danilov condition, depending (rationally) on a choice of a flag in \mathfrak{g} . Other choices of complement maps, encompassing Morelli’s construction, are studied in [9].

In [4] Brion and Vergne express the Todd class in *equivariant cohomology*, in the case when the associated fan is simplicial, as a linear combination of products of the canonical invariant divisors of the toric variety. These divisors generate the equivariant cohomology as an algebra.

In [5], Brylinski and Zhang give a formula for the equivariant Todd class of a general complete toric variety, as an element of a localized equivariant *homology* module. The Todd class is written as a linear combination of the homology classes defined by fixed points, the coefficients being germs of *meromorphic* functions on \mathfrak{g} . Their result can be recovered from our formula (1) by using the relations between the equivariant homology classes of orbit closures $[\bar{O}_\sigma]$, [5, Lemma 8.8]. For instance, in the case of the projective line $\mathbb{P}_1(\mathbb{C})$, one has the relation

$$-\xi[\mathbb{P}_1(\mathbb{C})] = [p_+] - [p_-].$$

From this relation and (2), we recover the formula of Brylinski–Zhang for $\mathbb{P}_1(\mathbb{C})$:

$$\text{Todd}(\mathbb{P}_1(\mathbb{C}))(\xi) = \frac{1}{1 - e^\xi} [p_+] + \frac{1}{1 - e^{-\xi}} [p_-].$$

In order to compute the Todd class, we start with the results of [4]. In the case of a simplicial variety, we rewrite the Todd class in cohomology using our Euler–Maclaurin formula of [1], then we translate it in homology using equivariant Poincaré duality for a simplicial toric variety. Finally, for the general case, we use the push-forward method in equivariant homology devised in [5].

When the fan Σ is dual to an integral polytope \mathfrak{p} , formula (1) is a rewriting, via Riemann–Roch, of the local Euler–Maclaurin expansion obtained in [1] for the sum $\sum_{x \in \mathfrak{p} \cap \mathbb{Z}^d} h(x)$ of the values $h(x)$ at integral points of \mathfrak{p} , where $h(x)$ is a polynomial function on \mathbb{R}^d . This sum is written as a sum of integrals over the various faces of \mathfrak{p} ,

$$\sum_{x \in \mathfrak{p} \cap \mathbb{Z}^d} h(x) = \sum_{\mathfrak{f} \in \mathcal{F}(\mathfrak{p})} \int_{\mathfrak{f}} D(\mathfrak{p}, \mathfrak{f}) \cdot h \tag{3}$$

where, for each face \mathfrak{f} of \mathfrak{p} , $D(\mathfrak{p}, \mathfrak{f})$ is the differential operator (of infinite order) with constant coefficients on \mathbb{R}^d whose symbol is the function $\mu(\mathfrak{t}(\mathfrak{p}, \mathfrak{f}))$ associated to the transverse cone of \mathfrak{p} along \mathfrak{f} . In dimension one, the partner of (2) is the classical Euler–Maclaurin summation formula for a lattice interval $[a_1, a_2]$,

$$\sum_{a_1}^{a_2} h(x) = \int_{a_1}^{a_2} h(t) dt - \sum_{n \geq 1} \frac{b(n)}{n!} h^{(n-1)}(a_1) + \sum_{n \geq 1} (-1)^n \frac{b(n)}{n!} h^{(n-1)}(a_2).$$

Let us note that the combinatorial formula (3) of [1] holds for all rational polytopes, not only integral ones.

Articles on toric varieties often stress their application to lattice points in polytopes. In this study, the relationship goes the other way round, a situation not uncommon in the interaction between science and technology: a progress in applications leads to a theoretical progress.

2. Definitions and notations

2.1. Cones

Let V be a rational vector space, that is to say a finite dimensional real vector space with a lattice denoted by Λ_V or simply Λ . We denote by V^* the dual space of V . We will denote elements of V by Latin letters x, y, v, \dots and elements of V^* by Greek letters ξ, \dots . We denote the duality bracket by $\langle \xi, x \rangle$.

V^* is equipped with the dual lattice Λ^* of Λ :

$$\Lambda^* = \{ \xi \in V^*; \langle \xi, x \rangle \in \mathbb{Z} \text{ for all } x \in \Lambda \}.$$

If S is a subset of V , we denote by $\text{lin}(S)$ the vector subspace of V generated by S and by S^\perp the subspace of V^* orthogonal to S :

$$S^\perp = \{ \xi \in V^*; \langle \xi, x \rangle = 0 \text{ for all } x \in S \}.$$

A subspace W of V is called rational if $W \cap \Lambda$ is a lattice in W . If W is a rational subspace, the image of Λ in V/W is a lattice in V/W , so that V/W is a rational vector space.

The space V , with lattice Λ , has a canonical Lebesgue measure, for which V/Λ has measure 1.

The set of nonnegative real numbers is denoted by \mathbb{R}_+ . A convex rational cone c in a rational space is a closed convex cone $\sum_{i=1}^k \mathbb{R}_+ v_i$ which is generated by a finite number of lattice vectors. In this article, we simply say cone instead of convex rational cone.

A cone c is called simplicial if it is generated by linearly independent primitive vectors v_1, \dots, v_k . The multiplicity $\text{mult}(c)$ of the simplicial cone c is the cardinal of the set $\Lambda \cap \text{lin}(c) / \sum_{i=1}^k \mathbb{Z} v_i$. It is equal to $|\det(v_1, \dots, v_k)|$ where the determinant is computed with respect to a basis of the lattice $\Lambda \cap \text{lin}(c)$.

The set of faces of a cone c is denoted by $\mathcal{F}(c)$. If c is pointed, then its vertex 0 is its unique face of dimension 0, while c is the unique face of maximal dimension $\dim c$.

If f is a face of the cone c , the *transverse cone of c along f* is the image of c in the quotient space $V/\text{lin}(f)$. We denote it by $t(c, f)$.

Let c be a cone in V . The dual cone c^* of c is the set of $\xi \in V^*$ such that $\langle \xi, x \rangle \geq 0$ for any $x \in c$.

We shall make use of subdivisions of cones.

Definition 1. A subdivision of a cone c is a finite collection \mathcal{C} of cones in $\text{lin}(c)$ such that:

- (a) The faces of any cone in \mathcal{C} are in \mathcal{C} .
- (b) If \mathfrak{d}_1 and \mathfrak{d}_2 are two elements of \mathcal{C} , then the intersection $\mathfrak{d}_1 \cap \mathfrak{d}_2$ is a face of both \mathfrak{d}_1 and \mathfrak{d}_2 .
- (c) We have $c = \bigcup_{\mathfrak{d} \in \mathcal{C}} \mathfrak{d}$.

The subdivision is called simplicial if it consists of simplicial cones.

Example 2. The basic example is the subdivision $\{\mathbb{R}_+, \mathbb{R}_-, \{0\}\}$ of the one-dimensional cone \mathbb{R} where \mathbb{R}_- denotes the opposite cone $-\mathbb{R}_+$.

2.2. A holomorphic function associated to a cone

We now recall the construction of [1]. Let V be a rational space with lattice Λ_V . We fix a scalar product $Q(x, y)$ on V . We assume that Q is rational, meaning that $Q(x, y)$ is rational for $x, y \in \Lambda_V$. To any cone c in a quotient space W of V , we associate in [1] a (germ at 0 of) holomorphic function on W^* , denoted by $\mu(c)(\xi)$.

The functions $\mu(c)(\xi)$ satisfy the following *local Euler–Maclaurin formula*:

$$S(c)(\xi) = \sum_{f \in \mathcal{F}(c)} \mu(t(c, f))(\xi) I(f)(\xi), \tag{4}$$

for $\xi \in W^*$ small.

This formula requires some explanations (see [1] for details).

The function $S(c)(\xi)$ is defined as a meromorphic function on W^* as follows. First, if the cone c is pointed, the series $\sum_{x \in c \cap \Lambda_W} e^{\langle \xi, x \rangle}$ converges only for those $\xi \in W^*$ such that $\langle \xi, x \rangle < 0$ for all $x \in c \setminus \{0\}$. The sum can be extended to a meromorphic function on W^* . This is $S(c)(\xi)$. If c is not pointed, the sum converges nowhere, and $S(c)(\xi) \equiv 0$ by definition.

The integral $I(f)(\xi) := \int_f e^{\langle \xi, x \rangle} dm_f(x)$ is defined as a rational function on W^* in a similar way. The measure $dm_f(x)$ is the normalized Lebesgue measure on $\text{lin}(f)$ defined by the lattice $\Lambda_W \cap \text{lin}(f)$.

In formula (4), the transverse cone $t(c, f)$ is a cone in the quotient space $W/\text{lin}(f)$. The function $\mu(t(c, f))$ is a function on a neighborhood of 0 in the dual $(W/\text{lin}(f))^* \cong \text{lin}(f)^\perp \subset W^*$. We give a sense to $\mu(t(c, f))(\xi)$ for a small $\xi \in W^*$ by extending this function to a neighborhood of 0 in the space W^* itself by means of orthogonal projection $W^* \rightarrow \text{lin}(f)^\perp$.

We have

$$\mu(\{0\}) = 1 \quad \text{and} \quad \mu(\mathbb{R}_+)(\xi) = \frac{1}{1 - e^\xi} + \frac{1}{\xi}.$$

Remark 3. In [1] the function $\mu(a)$ is defined for any rational *affine* cone a . Here we will need only the case of a cone with vertex 0.

Remark 4. The sum $S(c)$ is easy to compute when c is a simplicial cone in V of dimension equal to $d = \dim V$. Let v_1, v_2, \dots, v_d be the primitive vectors of the edges of c . We denote by $\square(c) = \sum_{i=1}^d [0, 1[v_i$ the semi-open parallelepiped generated by the v_i 's. Then

$$S(c)(\xi) = \left(\sum_{x \in \square(c) \cap \Lambda_V} e^{\langle \xi, x \rangle} \right) \prod_{i=1}^d \frac{1}{1 - e^{\langle \xi, v_i \rangle}}. \tag{5}$$

In particular, the function $\xi \mapsto \prod_{i=1}^d \langle \xi, v_i \rangle S(c)(\xi)$ is holomorphic near 0 in V^* .

Recall the relation between the rational function $I(c)$ and the rational functions $I(f)$ where f varies over the set of facets of c .

Let c be a pointed cone in V such that $\text{lin}(c) = V$. Let c^* be its dual cone in V^* , and let $\mathcal{E}(c^*)$ be the set of its edges. An edge τ of c^* defines the facet $f_\tau = \text{lin}(\tau)^\perp \cap c$ of c . We choose primitive generators n_τ on each edge τ .

Lemma 5. For $\xi \in V^*$ and $v \in V$, we have

$$-\langle v, \xi \rangle I(c)(\xi) = \sum_{\tau \in \mathcal{E}(c^*)} \langle v, n_\tau \rangle I(f_\tau)(\xi).$$

Proof. Let us denote by λ the Lebesgue volume form on V defined by the lattice and an orientation. We have, for c oriented accordingly,

$$-\langle v, \xi \rangle I(c)(\xi) = - \int_c e^{\langle \xi, x \rangle} \langle v, \xi \rangle \lambda = - \int_c d\alpha,$$

where the differential form α on V is given by

$$\alpha = e^{\langle \xi, x \rangle} \iota_v \lambda.$$

Then Lemma 5 is just Stokes formula. \square

2.3. Todd measure of a cone

Let $G \simeq (\mathbb{C}^*)^d$ be a complex torus. We denote its character group by Λ , the real vector space spanned by Λ by \mathfrak{g}^* , the dual of \mathfrak{g}^* by \mathfrak{g} and the dual lattice by $\Lambda^* \subset \mathfrak{g}$.

We denote by $S(\mathfrak{g}^*)$ the symmetric algebra over \mathfrak{g}^* , identified with the algebra of polynomial functions on \mathfrak{g} and by $\hat{S}(\mathfrak{g}^*)$ its completion, the algebra of formal power series. If h is a holomorphic function on \mathfrak{g} defined near 0, its Taylor series is an element of $\hat{S}(\mathfrak{g}^*)$ also denoted by h .

In the following, we will apply the construction of the preceding Section 2.2 with \mathfrak{g}^* in the role of the space V , therefore the dual V^* will be the space \mathfrak{g} .

Let $\sigma \subset \mathfrak{g}$ be a cone. The dual cone σ^* of σ is the set of $x \in \mathfrak{g}^*$ such that $\langle \xi, x \rangle \geq 0$ for any $\xi \in \sigma$. The vector subspace $\text{lin}(\sigma)^\perp \subset \mathfrak{g}^*$ of elements $x \in \mathfrak{g}^*$ such that $\langle \xi, x \rangle = 0$ for any $\xi \in \sigma$ is contained in σ^* .

Definition 6. Let σ be a cone in \mathfrak{g} . We denote by $\check{\sigma}$ the image of σ^* in the quotient space $\mathfrak{g}^*/\text{lin}(\sigma)^\perp$.

The dual of $\mathfrak{g}^*/\text{lin}(\sigma)^\perp$ is identified with the subspace $\text{lin}(\sigma) \subset \mathfrak{g}$.

Definition 7. $\mu^*(\sigma)$ is the (germ at 0 of) holomorphic function on \mathfrak{g} defined as follows. For $\xi \in \text{lin}(\sigma)$,

$$\mu^*(\sigma)(\xi) := \mu(\check{\sigma})(\xi),$$

then $\mu^*(\sigma)$ is extended to \mathfrak{g} by orthogonal projection.

A crucial property of these functions is that the assignment $\sigma \mapsto \mu^*(\sigma)(\xi)$ is a *simple valuation*:

Theorem 8. (See [1, Corollary 21].) Let σ_0 be a cone in \mathfrak{g} , and let \mathcal{C} be a subdivision of σ_0 . Then

$$\mu^*(\sigma_0) = \sum_{\sigma \in \mathcal{C}; \dim \sigma = \dim \sigma_0} \mu^*(\sigma).$$

Thus, extending a definition of [9], we may call $\mu^*(\sigma)(\xi)$ the *equivariant Todd measure of σ* .

2.4. The G -toric variety defined by Σ

Let $G \simeq (\mathbb{C}^*)^d$ be a complex torus. Let Σ be a G -fan. For $1 \leq i \leq d$, we denote by $\Sigma[i]$ the set of cones in Σ of dimension i .

The G -toric variety defined by Σ [6,7] is denoted by X_Σ . Recall that a cone σ in Σ defines an open G -invariant affine subvariety U_σ of X_Σ . In this article, we follow the conventions of [4]. Thus, if $r \in \sigma^* \cap \Lambda$, then r is a regular function on U_σ of weight r with respect to the action of G on X_Σ . This convention determines the action of G on X_Σ . Note that this convention is contrary to that of [3].

The open set U_σ contains a unique relatively closed G -orbit O_σ whose closure in X_Σ is denoted by \bar{O}_σ . If $\sigma \in \Sigma[d]$, then U_σ contains a unique G -fixed point p_σ and $O_\sigma = \{p_\sigma\}$.

Example 9. Let $G = \mathbb{C}^*$. Denote by ℓ the canonical generator of Λ ($\ell(u) = u$, for $u \in \mathbb{C}^*$). The projective line $P_1(\mathbb{C})$ is associated to the fan, $\mathbb{R}_-, \{0\}, \mathbb{R}_+$ in \mathfrak{g} . The fixed points are p_+ and p_- , the action of G on the tangent space at p_+ being given by $-\ell$, and the action of G on the tangent space at p_- being given by ℓ . Then

$$\bar{O}_{\{0\}} = P_1(\mathbb{C}), \quad \bar{O}_{\mathbb{R}_+} = \{p_+\}, \quad \bar{O}_{\mathbb{R}_-} = \{p_-\}.$$

Denote by R_Σ the algebra of continuous *piecewise polynomial* functions on $\bigcup_{\sigma \in \Sigma} \sigma$. An element of R_Σ is a function f such that, for any cone σ in $\Sigma[d]$, the restriction of f to σ is equal to the restriction to σ of an element of $S(\mathfrak{g}^*)$ which we denote by $f|_\sigma$.

R_Σ is a module over $S(\mathfrak{g}^*)$. We define $\hat{R}_\Sigma = \hat{S}(\mathfrak{g}^*) \otimes_{S(\mathfrak{g}^*)} R_\Sigma$. If $\sigma \in \Sigma[d]$, the restriction map $f \mapsto f|_\sigma$ extends to a map from \hat{R}_Σ to $\hat{S}(\mathfrak{g}^*)$.

Let $H_G^*(X_\Sigma)$ denote the equivariant homology group of X_Σ . It is a module over $S(\mathfrak{g}^*)$, with the following relations (see [3]). Any G -invariant closed subvariety Y of X_Σ defines an element of $H_G^*(X_\Sigma)$, denoted by $[Y]$. Let ν be an element of Λ , that is a character of G . If f is a function on Y of weight ν , then $-\nu[Y] = [\text{div}(f)]$.

Example 10. Return to Example 9. Then $-\ell[P_1(\mathbb{C})] = [p_+] - [p_-]$.

Indeed, ℓ is a rational function f on $P_1(\mathbb{C})$ of weight ℓ and $\text{div}(f) = \{p_+\} - \{p_-\}$.

3. A formula for the equivariant Todd class of a complete simplicial toric variety

In this section we assume that Σ is a complete simplicial fan. Then X_Σ is complete and is locally the quotient of a smooth variety by a finite group. One says that X_Σ is a simplicial complete toric variety.

Let $H_G^*(X_\Sigma)$ be the equivariant cohomology algebra of X_Σ . The Todd class $\text{Todd}_{H_G^*}(X_\Sigma)$ is an element of the completed algebra $\hat{H}_G^*(X_\Sigma)$, where

$$\hat{H}_G^*(X_\Sigma) = \hat{S}(\mathfrak{g}^*) \otimes_{S(\mathfrak{g}^*)} H_G^*(X_\Sigma).$$

Let us recall Brion’s isomorphism between $H_G^*(X_\Sigma)$ and R_Σ .

Let $\alpha \in H_G^*(X_\Sigma)$. For every $\sigma \in \Sigma[d]$, the restriction of α to the fixed point p_σ is an element of $S(\mathfrak{g}^*)$ which we denote by $\alpha(p_\sigma)$.

Theorem 11. (See [2].) *There exists a unique algebra isomorphism B of $H_G^*(X_\Sigma)$ with R_Σ such that, for $\alpha \in H_G^*(X_\Sigma)$ and $\sigma \in \Sigma[d]$, one has $B(\alpha)|_\sigma = \alpha(p_\sigma)$.*

This isomorphism B extends to an isomorphism from $\hat{H}_G^*(X_\Sigma)$ to \hat{R}_Σ .

We recall the formula [4, 4.1, Theorem] which expresses the Todd class $\text{Todd}_{H_G^*}(X_\Sigma)$ of X_Σ as an element of \hat{R}_Σ .

Let $\sigma \in \Sigma[d]$. The dual cone σ^* is a simplicial cone of dimension d in \mathfrak{g}^* . Let v_i be the primitive generators of the edges of σ^* .

$$B(\text{Todd}_{H_G^*}(X_\Sigma))|_\sigma(\xi) = \frac{1}{\text{mult}(\sigma^*)} \prod_{i=1}^d \langle -\xi, v_i \rangle S(\sigma^*)(\xi) \quad \text{for } \xi \in \mathfrak{g}. \tag{6}$$

Remark 12. By Remark 4, the right-hand side is indeed holomorphic at 0.

Let σ be any cone in Σ . Following [4, 3.3], we consider the following element φ_σ of R_Σ .

Let $\tau \in \Sigma[1]$ be a cone of dimension 1 and let η_τ be the generator of $\tau \cap \Lambda^*$. We denote by $\varphi_\tau \in R_\Sigma$ the continuous piecewise linear function on \mathfrak{g} such that $\varphi_\tau(\eta_\tau) = 1$ and $\varphi_\tau(\eta_{\tau'}) = 0$ for all $\tau' \in \Sigma[1]$ different from τ .

Definition 13. Let σ be a cone in Σ and $\text{mult}(\sigma)$ its multiplicity. Let $\mathcal{E}(\sigma)$ be the set of edges of σ . Then we set:

$$\varphi_\sigma = \text{mult}(\sigma) \prod_{\tau \in \mathcal{E}(\sigma)} \varphi_\tau.$$

The function φ_σ vanishes identically on all cones which do not contain σ .

We compute the restriction $\varphi_\sigma|_{\sigma_0}$ for a cone σ_0 of dimension d . If σ_0 does not contain σ , then $\varphi_\sigma|_{\sigma_0} = 0$. Otherwise, let $\sigma_0 \in \Sigma[d]$ containing σ , let $\eta_1^{\sigma_0}, \dots, \eta_d^{\sigma_0}$ be the primitive generators of the edges of σ_0 such that $\eta_1^{\sigma_0}, \dots, \eta_r^{\sigma_0}$ generate σ , and let $v_1^{\sigma_0}, \dots, v_d^{\sigma_0}$ be the primitive generators of σ_0^* , indexed in such a way that $\langle v_i^{\sigma_0}, \eta_i^{\sigma_0} \rangle > 0$. Let \mathfrak{f} be the cone in \mathfrak{g}^* generated by $v_{r+1}^{\sigma_0}, \dots, v_d^{\sigma_0}$. Thus $\text{lin}(\mathfrak{f}) = \text{lin}(\sigma)^\perp$.

Lemma 14.

$$\varphi_\sigma|_{\sigma_0} = \frac{\text{mult}(\mathfrak{f})}{\text{mult}(\sigma_0^*)} \prod_{j=1}^r v_j^{\sigma_0}.$$

Proof. Let $q_j = \langle v_j^{\sigma_0}, \eta_j^{\sigma_0} \rangle$, so that the dual basis to the basis $\{\eta_j^{\sigma_0}\}_{j=1}^d$ consists of the vectors $\{\frac{1}{q_j} v_j^{\sigma_0}\}_{j=1}^d$. The definition of φ_σ implies that

$$\varphi_\sigma|_{\sigma_0} = \text{mult}(\sigma) \prod_{j=1}^r \frac{1}{q_j} v_j^{\sigma_0}.$$

Thus we need to prove:

$$\frac{\text{mult}(\mathfrak{f})}{\text{mult}(\sigma_0^*)} = \text{mult}(\sigma) \prod_{j=1}^r \frac{1}{q_j}.$$

This equality is obtained immediately by computing the multiplicities as absolute values of determinants, using a basis of Λ in which the matrix of the vectors η_1, \dots, η_d is upper triangular. \square

As Σ is a complete simplicial fan, the Poincaré isomorphism $P : H_*^G(X_\Sigma) \mapsto H_G^*(X_\Sigma)$ is an isomorphism of $S(\mathfrak{g}^*)$ -modules between the equivariant homology and the equivariant cohomology.

Proposition 15. (See [4, 3.3].)

$$B(P(\bar{O}_\sigma)) = (-1)^{\dim \sigma} \varphi_\sigma.$$

The classes φ_σ , when σ varies over all cones in Σ , generates R_Σ as a module over $S(\mathfrak{g}^*)$. Thus the equivariant Todd class $\text{Todd}_{H_G^*}(X_\Sigma)$ of a complete simplicial toric variety can be expressed as a combination of the classes φ_σ with coefficients in $\hat{S}(\mathfrak{g}^*)$. Such an expression is highly not unique as there are relations between the elements φ_σ . From now on, we choose a rational scalar product on \mathfrak{g}^* . Given this choice, we are able to give a canonical formula.

Proposition 16. Let G be a torus with Lie algebra \mathfrak{g} . Let Σ be a complete rational simplicial fan in \mathfrak{g} and let X_Σ be the corresponding G -toric variety. Choose a scalar product on \mathfrak{g} . Then the equivariant Todd class $\text{Todd}_{H_G^*}(X_\Sigma)$ of X_Σ is given by the following combination of the classes φ_σ with coefficients in $\hat{S}(\mathfrak{g}^*)$:

$$B(\text{Todd}_{H_G^*}(X_\Sigma)) = \sum_{\sigma \in \Sigma} (-1)^{\dim \sigma} \mu^*(\sigma) \varphi_\sigma. \tag{7}$$

Example 17. Return to Example 9. The restriction of the Todd class to \mathbb{R}_+ is $\frac{-\xi}{1-e^\xi}$. Compute the restriction to \mathbb{R}^+ of the right-hand side of formula (7):

$$\mu^*({0})\varphi_{\{0\}} - \mu^*(\mathbb{R}_+)\varphi_{\mathbb{R}_+} - \mu^*(\mathbb{R}_-)\varphi_{\mathbb{R}_-}.$$

We obtain

$$1 - \xi \left(\frac{1}{1 - e^\xi} + \frac{1}{\xi} \right) = \frac{-\xi}{1 - e^\xi}.$$

In the same way, we check the equality on \mathbb{R}_- .

Proof. We only need to check that both members of Eq. (7) agree on each cone $\sigma_0 \in \Sigma[d]$. Let $\eta_1, \eta_2, \dots, \eta_d$ be the primitive generators of the edges of σ_0 . Let v_1, v_2, \dots, v_d be the primitive generators of the edges of σ_0^* , so that $\langle v_i, \eta_i \rangle > 0$. As φ_σ vanishes if σ is not contained in σ_0 , the right-hand side of Eq. (7) restricted to σ_0 is a sum over the faces of σ_0 . For any subset J of $\{1, \dots, d\}$, we denote by $\sigma_J \subset \sigma_0$ the cone generated by the vectors η_j for $j \in J$. The cone $\{0\}$ corresponds to the empty set.

Let us write the local Euler–Maclaurin formula (4) for the cone σ_0^* . We label the faces f_J of σ_0^* by the subsets $J \subset \{1, \dots, d\}$, by setting $f_J = \sum_{j \notin J} \mathbb{R}_+ v_j$. Thus, for each subset J , we have $\text{lin}(f_J) = \text{lin}(\sigma_J)^\perp$. The transverse cone $t(\sigma_0^*, f_J)$ and the projected cone $\check{\sigma}_J \subset \mathfrak{g}^* / \text{lin}(\sigma_J)^\perp$ coincide. The integral $I(f_J)(\xi)$ is given by

$$I(f_J)(\xi) = \text{mult}(f_J) \prod_{j \notin J} \frac{1}{\langle -\xi, v_j \rangle}.$$

We obtain

$$S(\sigma_0^*)(\xi) = \sum_J \text{mult}(f_J) \mu(\check{\sigma}_J)(\xi) \prod_{j \notin J} \frac{1}{\langle -\xi, v_j \rangle}. \tag{8}$$

Thus

$$\begin{aligned} \frac{1}{\text{mult}(\sigma_0^*)} \prod_{1 \leq j \leq d} \langle -\xi, v_j \rangle S(\sigma_0^*)(\xi) &= \sum_J \frac{\text{mult}(f_J)}{\text{mult}(\sigma_0^*)} \mu(\check{\sigma}_J)(\xi) \prod_{j \in J} \langle -\xi, v_j \rangle \\ &= \sum_J (-1)^{|J|} \mu(\check{\sigma}_J)(\xi) \frac{\text{mult}(f_J)}{\text{mult}(\sigma_0^*)} \prod_{j \in J} \langle \xi, v_j \rangle \\ &= \sum_J (-1)^{|J|} \mu(\check{\sigma}_J)(\xi) \varphi_{\sigma_J}(\xi). \end{aligned}$$

The last equality follows from Lemma 14. Hence, recalling Eq. (6), we have proven Proposition 16. \square

4. A formula for the Todd class of a complete toric variety

In this section we prove the formula (1) announced in the introduction, in the case of any complete toric variety, not necessary simplicial.

Let Σ be a complete fan in \mathfrak{g} . We do not assume that Σ is simplicial. The equivariant Todd class $\text{Todd}_{H_*^G}(X_\Sigma)$ of X_Σ is then defined as an element of the completed equivariant homology module $\hat{H}_*^G(X_\Sigma)$,

$$\hat{H}_*^G(X_\Sigma) = \hat{S}(\mathfrak{g}^*) \otimes_{S(\mathfrak{g}^*)} H_*^G(X_\Sigma).$$

This module is generated over $\hat{S}(\mathfrak{g}^*)$ by the elements $[\bar{O}_\sigma]$ for all $\sigma \in \Sigma$. There are relations between these generators. However, given a scalar product on \mathfrak{g} , we obtain $\text{Todd}_{H_*^G}(X_\Sigma)$ as a combination of the classes $[\bar{O}_\sigma]$, with canonical coefficients in $\hat{S}(\mathfrak{g}^*)$.

Theorem 18. *Let G be a torus with Lie algebra \mathfrak{g} . Let Σ be a complete rational fan in \mathfrak{g} and let X_Σ be the corresponding G -toric variety. For $\sigma \in \Sigma$, let \bar{O}_σ be the corresponding G -invariant subvariety of X_Σ , and let $[\bar{O}_\sigma]$ be its class in the equivariant homology ring $H_*^G(X_\Sigma)$. Choose a scalar product on \mathfrak{g} . Then the equivariant Todd class $\text{Todd}_{H_*^G}(X_\Sigma)$ of X_Σ is given by the following combination of the elements $[\bar{O}_\sigma]$:*

$$\text{Todd}_{H_*^G}(X_\Sigma) = \sum_{\sigma \in \Sigma} \mu^*(\sigma) [\bar{O}_\sigma]. \tag{9}$$

Proof. We use the push-forward argument of [5].

When Σ is a complete simplicial fan, the cohomological Todd class is the Poincaré dual of the homological Todd class. Thus Theorem 18 follows right away from Propositions 16 and 15.

For the general case, we take advantage of the functoriality of the Todd genus and use a refinement of the fan Σ , in order to reduce the proof to the case of a complete simplicial toric variety. The valuation property (Theorem 8) will be crucial.

The cones $\sigma \in \Sigma$ can be subdivided, yielding a fan $\tilde{\Sigma}$, in such a way that all the cones in $\tilde{\Sigma}$ are simplicial. Let \tilde{X} be the toric variety corresponding to the fan $\tilde{\Sigma}$. Then \tilde{X} is complete simplicial, and there is a proper birational map $f: \tilde{X} \rightarrow X_\Sigma$. The equivariant Todd genus of X_Σ coincides with the push-forward of the equivariant Todd genus of \tilde{X} . Moreover, let $\tilde{\sigma}$ be a cone in $\tilde{\Sigma}$, let $\tilde{O}_{\tilde{\sigma}}$ be the corresponding subvariety of \tilde{X} , and let σ be the smallest cone of Σ containing $\tilde{\sigma}$. Then $f_*([\tilde{O}_{\tilde{\sigma}}]) = [\tilde{O}_\sigma]$ if $\tilde{\sigma}$ and σ have the same dimension, and $f_*([\tilde{O}_{\tilde{\sigma}}]) = 0$ otherwise.

For the variety \tilde{X} , we have

$$\text{Todd}_{H_*^G}(\tilde{X}) = \sum_{\tilde{\sigma} \in \tilde{\Sigma}} \mu^*(\tilde{\sigma})[\tilde{O}_{\tilde{\sigma}}]. \tag{10}$$

Let $\sigma \in \Sigma$. The cones $\tilde{\sigma} \in \tilde{\Sigma}$ which are contained in σ form a subdivision of σ . Therefore we have by Theorem 8,

$$\mu^*(\sigma) = \sum_{\{\tilde{\sigma} \in \tilde{\Sigma}, \tilde{\sigma} \subset \sigma, \dim \tilde{\sigma} = \dim \sigma\}} \mu^*(\tilde{\sigma}).$$

Thus we obtain (9) by taking the push-forward of both sides of (10). \square

By localizing Eq. (9), we recover the formula of Brylinski–Zhang [5, Theorem 9.4], in the following corollary.

We denote by L the multiplicative subset of $S(\mathfrak{g}^*)$ which consists of products $\prod_{i \in I} v_i$ of nonzero elements of \mathfrak{g}^* . We denote by $L^{-1}H_*^G(X_\Sigma)$ the corresponding localized module. It is generated by the classes of the fixed points p_{σ_0} , for $\sigma_0 \in \Sigma[d]$. More precisely we have the following relations

Lemma 19. *Let σ be any cone of Σ . In $L^{-1}H_*^G(X_\Sigma)$, we have*

$$[\tilde{O}_\sigma] = \sum_{\{\sigma_0 \in \Sigma[d], \sigma \in \mathcal{F}(\sigma_0)\}} I(\sigma_0^* \cap \text{lin}(\sigma)^\perp)[p_{\sigma_0}].$$

Proof. The proof is by induction on the codimension of σ . If σ is of dimension d , there is nothing to prove. Otherwise, take a nonzero $v \in \text{lin}(\sigma)^\perp$. By Lemma 8.8 of [5], we have

$$-v[\tilde{O}_\sigma] = \sum_{\{\tau \in \Sigma, \sigma \in \mathcal{F}'(\tau)\}} \langle \eta(\sigma, \tau), v \rangle [\tilde{O}_\tau]$$

where $\mathcal{F}'(\tau)$ denotes the set of facets of τ and $\eta(\sigma, \tau)$ is the generator of the semigroup $\tau \cap \Lambda / \sigma \cap \Lambda$.

Applying the induction hypothesis to each τ , we have to prove, for a fixed $\sigma_0 \in \Sigma[d]$,

$$-vI(\sigma_0^* \cap \text{lin}(\sigma)^\perp) = \sum_{\{\tau \in \mathcal{F}(\sigma_0), \sigma \in \mathcal{F}'(\tau)\}} \langle \eta(\sigma, \tau), v \rangle I(\sigma_0^* \cap \text{lin}(\tau)^\perp). \tag{11}$$

When τ runs through the set of faces of σ_0 having σ as a facet, the cone $\sigma_0^* \cap \text{lin}(\tau)^\perp$ runs through the set of facets of $\sigma_0^* \cap \text{lin}(\sigma)^\perp$. Thus relation (11) follows from Lemma 5. \square

Corollary 20. In $L^{-1} \hat{H}_*^G(X_\Sigma)$ the Todd class is given by

$$\text{Todd}_{H_*^G}(X_\Sigma) = \sum_{\sigma_0 \in \Sigma[d]} S(\sigma_0^*)[p_{\sigma_0}]. \tag{12}$$

Proof. We rewrite (9) using Lemma 19 and reverse the summations. Let $\sigma_0 \in \Sigma[d]$. The faces f of σ_0^* are in one-to-one correspondence with the faces σ of σ_0 , with $f = \sigma_0^* \cap \text{lin}(\sigma)^\perp$ corresponding to σ . Moreover, the transverse cone $t(\sigma_0^*, f)$ is equal to $\check{\sigma}$. We thus obtain

$$\text{Todd}_{H_*^G}(X_\Sigma) = \sum_{\sigma_0 \in \Sigma[d]} \sum_{f \in \mathcal{F}(\sigma_0)} \mu(t(\sigma_0^*, f)) I(f)[p_{\sigma_0}].$$

Hence the corollary follows from Euler–Maclaurin expansion (4) of $S(\sigma_0^*)$. \square

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